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THE MATHEMATICS TEACHER

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THE MATHEMATICS TEACHER

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THE MATHEMATICS TEACHER

Volume XL Number 5

Edited by William David Reeve

A Philosophy for the Mathematics Teacher

By C. V. Newsom
Oberlin College, Oberlin, Ohio

We who are teachers of mathematics are proud. Our ancestors in the profession include some of the greatest teachers in history. We are direct descendents of Pythagoras, Euclid, and the great teachers of Plato's Academy. The students we and our forebears have trained have made history; they have assisted in the development of methods and devices that have been responsible for the high level to which man has attained.

The subject we teach is the best organized of all the disciplines. Except possibly for a few of the outer fringes, the material of the various mathematical systems is undoubtedly consistent. Although mathematics is man-made, it has utilized those results of mind's activity that show man at his best. Mathematics permits one to deal with a chaotic universe, and make it seem orderly. The great triumphs of physical science have been, more than most people realize, mathematical triumphs. The complete record of mathematics in the last war, when it is available, will increase our pride. Certainly the teacher of mathematics can walk before a class with more confidence, actually with more self-respect, than can most teachers.

In the classroom, the teacher of mathematics is always conscious of the fact that he is teaching only a small fragment of one of the great mathematical systems. He should convey the same point of view to his students. Virtually every course has very little mathematical significance except as it is related to the mathematics that follows and, also, to that which has preceded. I have observed through the years that the grade school teacher is always anticipating the mathematics of high school; the high school teacher watches college entrance requirements and the needs of college mathematics with a wary eye; the college teacher seeks the approval of graduate schools where his students will go on for advanced degrees; and graduate schools train their students so that they will contribute to the future records of research compiled for each institution. To equalize this emphasis, I hope to see the day when all high school teachers will feel a responsibility for giving new meaning to the mathematics of the grades, when college teachers will attempt in an explicit manner to interpret and throw new light upon the mathematics of high school, when more research men in mathematics will assist in the development of better educational programs for the training of mathematical scholars.

The course in high school algebra should anticipate calculus, but what is even more important, it should reteach and reexplain arithmetic. Rather than condemn

the grade school teacher who passes students who are poorly prepared, the algebra teacher could spend an equivalent amount of energy putting that essential finishing touch to the students' arithmetical knowledge. Because of extensions to the number system necessary to the development of algebra, the numbers and operations of elementary arithmetic should take on greater significance when algebra is properly taught. The actual technique involved in the multiplication or division of two numbers of several digits becomes immediately understandable when the numbers are treated as algebraic polynomials in powers of ten. The special product formulas and the binomial theorem take on greater significance when related to various short-cut and approximate methods in arithmetic. The list may be extended indefinitely.

In geometry, there should be no hesitation in using algebraic demonstrations. Trigonometry can be taught in such a manner that concepts in both algebra and geometry receive clarification. One of the merits of a course in calculus is the opportunity given the teacher to review and summarize the mathematics that has preceded. Similar statements might be made about every course in mathematics.

In selecting the subject matter for a course, the good teacher of mathematics keeps in mind at least three points. First, the material must be so chosen that the resulting structure gives the impression of unity and completeness, and provides an adequate foundation for future extensions. The most common criticism that comes from my students is that much of their training in mathematics seems to have been concerned with a collection of apparently unrelated tricks and ideas. Modern attempts at unification within the field of elementary mathematics certainly deserve every encouragement. A modern writer who described unified mathematics as scrambled mathematics missed the point; nevertheless, such criticisms offer a challenge to those of us who believe in greater unification. In the second place, the utilitarian aspects of mathematics cannot be ignored in the classroom. Certainly there is great motivation for both students and teacher when a classroom presentation continually incorporates the fact that we are considering those methods which have made possible our great technological and scientific achievements. American mathematics has become too pure for its own good. I have found in college that even the most confirmed crusader for the liberal arts in a narrow sense still feels a sense of satisfaction and personal triumph when he learns comparatively simple mathematical features of the law of growth or Kepler's laws in astronomy. Sometimes, however, I am amazed that some teachers of mathematics have so much difficulty thinking of meaningful applications; this may indicate a serious deficiency in our training of teachers. As a third point, mathematics in common with every other subject in the curriculum should accept responsibility for providing those experiences that make for cultural breadth and appreciation. Perhaps this third point contains the other two as subdivisions.

I think that every mathematics teacher must believe and must continually emphasize that there is a literature of mathematics as important to man's cultural development as the great classics of poetry. The person commonly known as artist has no monopoly upon the aesthetic. It would take a person of quite limited horizons to fail to see the beauty in Euclid's proof that there exists an infinitude of primes or even in some of the demonstrations of the Pythagorean theorem. In spite of the fact that I insist that we must emphasize the practical, I affirm just as strongly that man does not live by bread alone. We owe an obligation to our students to give them some of the dessert as well as the bread. Of course, such an objective will never be apparent if the teacher's lack of enthusiasm conveys the impression that the part which should be the dessert is merely

composed of scraps that are left over. No person has a better opportunity than the mathematician to achieve a balance between the practical and the impractical; this should be reflected in his personality and in his attitude toward life. I think there is a very definite reason behind the finding of the psychologist, Henry C. Link, that mathematicians generally show a minimum of maladjustment and are rated at the top in happiness.

In presenting the material of mathematics the good teacher will always remember that rigor is a relative term. As Felix Klein so often pointed out, that which is best from a pedagogical point of view is not always the most logical by the teacher's standards. Even though mathematics, by definition, is a deductive science, its presentation to students and its natural development will be more inductive than deductive. Mathematicians are too sensitive on this score; an author of some important research will publish an elegant deductive demonstration of his results, but he feels it is unethical to reveal the long and laborious process, usually inductive, that has preceded the ultimate deductive organization. For too long we have made the mistake of plunging the student of plane geometry into a discipline highly organized along deductive lines. Of course, the deductive should be shown and studied but that type of organization reveals the subject matter of the course in its most advanced stage of development. By contrast, most elementary courses in algebra sin in the reverse direction by never showing any part of the ultimate logical organization which mathematicians find to be necessary.

Although it is not completely appropriate, I can not resist introducing at this point one of my favorite quotations from the literature of mathematics. It is from the pen of G. C. Evans and is hidden from most mathematicians in his book entitled *Mathematical Introduction to Economics*. Evans writes:

> The systematization which occurs in a theoretical science, as we may properly call it in order to distinguish it from a natural or applied science, is a process which is apt to come late in the development of a subject. Evidently some fields of knowledge are hardly ready for it, for it is typified by a free spirit of making hypotheses and definitions rather than a mere recognition of facts. But when we find this feeling for hypothesis and definition and, in addition, become involved in chains of deductive reasoning, we are driven to a characteristic method of construction and analysis which we may call the mathematical method. It is not a question as to whether mathematics is desirable or not in such a subject. We are in fact forced to adopt the mathematical method as a condition of further progress.

Every teacher must keep in mind that mathematics in a technical sense is a triumph of applied logic, but the devices of logic are not simple, they are not inherited, and the teacher of mathematics must take each student by the hand as he leads him through the complexities that are involved. As teachers, let us not forget that genuine mathematical understanding cannot precede the student's comprehension of the logical and his ability to read and understand.

A good teacher cannot afford to ignore the historical background of his subject. The biologists have their theory of recapitulation which states that to a great extent each individual in his early development reproduces in himself the early evolution of the race. I sincerely believe, more than we sometimes realize, that each student develops in his understanding of mathematics along much the same line that history reveals that man developed. It is apparent, of course, that too rigid an interpretation can be placed upon this statement, but I, as a teacher, must admit my own indebtedness to my study of man's historical struggle to build the remarkable system called mathematics. A knowledge of history is imperative for other reasons. The evolution of such a discipline as mathematics can not be separated from the rest of the stream of history; mathematics has been a contributor and a recipient in terms of the other factors of civilization. The fact that mathematics is a living body should dominate the attitudes of the

teacher and should be portrayed to our students.

In the same vein, it has always seemed reasonable to me that the person who has had direct experience in organizing and adding to our mathematical structure should have additional background for becoming a good teacher of mathematics; in general, I think this is true provided that the person in question is interested in teaching. There is plenty of opportunity for every mathematics teacher, irrespective of background, to become a creator in the field, and I think the experience is profitable.

The successful teacher of mathematics must make a definite attempt to become acquainted with trends in mathematical science and with the various ideas that are continually being advanced to improve our teaching of the subject. Any teacher having a background so inadequate that he is hesitant and uncertain cannot hope to win the confidence of his students. Let us not forget that Jesus of Nazareth impressed his listeners because He spoke as one who had authority. The teacher must always remain a student; teaching is no profession for one who is weak in spirit. Membership in professional societies, the reading of professional journals, and participation in professional discussions are all imperative if a teacher does not want to be characterized by the term "fossil." Sometimes, however, I believe that no profession comprises members with so little professional interest as our own; I should like to know the reason. If the membership in the National Council could be multiplied by ten, there would be a noticeable improvement in standards of teaching.

Finally, in spite of his pride, the teacher of mathematics is modest in his realization that his field is only one of the many important divisions of our total intellectual endeavor. In our zeal to push our own interests, it is easy to forget that the needs of modern man are indeed complicated; we must see our own field in its proper perspective in relation to other fields. The traditional curriculum is feeling the impact of the development of the social sciences, the arts, the crafts, and so on. All of these new disciplines have a proper place in our educational program. We must join with educational specialists in all fields, as we expect them to join with us, in the development of an educational effort that will have the greatest significance for the largest number of people. The quickest way for us as mathematicians to "lose face" with the public is to adopt an exaggerated point of view in regard to our importance; this does not mean, however, that we should not insist upon a rightful place for our subject. We must recognize individual differences in mental capacity, in interest, and in future accomplishment. The teacher of mathematics not only wants to train the research men of the future but he wants to be a part of the movement that will give our country better citizens.

Out of the educational controversies that are raging, something substantial will come; you see I remain an optimist. If we as teachers approach the field of education with understanding, and properly evaluate our own relation to it, we should have no fears for our profession. The American public wants good education, and parents will never agree to an inferior status for any part of the curriculum that is essential to the future success of their children. The years ahead will be brilliant years for us; there is no reason why we as mathematicians should not continue to show the pride in our field that has been our inheritance from the past.

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Mathematics for Responsible Citizenship

By HAROLD P. FAWCETT
The Ohio State University, Columbus, Ohio

DURING recent years, we have seen the peoples of the earth engaged in the mobilization of their material, intellectual, and spiritual resources for a mortal struggle between two opposing philosophies of government. We have heard the rumble of roaring tanks, the hum of many bombers, and the tread of marching feet; but above these symbols of military might we have heard the steady and persistent call for young men and women who are mathematically literate. The demands of modern citizenship have revealed with impressive effect that proficiency in the field of mathematics is a powerful military asset in a world that is girded for war. The guns are now silent. The bombers are quiet. The concept of one world echoes vaguely but unmistakably through the council chambers of the universe and against this background I wish to present the proposition that mathematical literacy is likewise a powerful asset for responsible citizenship in a world that is planning for peace.

A respected body of evidence can be mobilized in support of this proposition. There is, for example, the splendid report of the Joint Commission which considers "The Place of Mathematics in Secondary Education." There is also the report of the Committee on the Function of Mathematics in General Education and the second report of the Post-War Commission includes many recommendations concerning our obligation to insure "mathematical competence for the ordinary affairs of life." Nor would I overlook the effective statements which have come from the skillful pens of such distinguished authors as David Eugene Smith, Ear, Raymond Hedrick, William David Reevel W. S. Schlauch, William Betz, Sir Cyril Ashford, and W. Leitzman. There is general agreement in all of these pronouncements that the fabric of modern society is permeated with mathematical concepts and together they provide convincing support of the proposition already enunciated. It is in fact presumptuous to assume that I might add anything of importance to the discussion of this significant problem but at the risk of being so labeled, I would point out that in each of these reports there are significant implications for classroom practice to which we must be persistently faithful if the teaching of mathematics is to make its greatest contribution to the development of responsible citizenship.

Let us recognize in the first place that "responsible citizenship" is a variable and that if we are to contribute to this desirable outcome through the experiences provided in our classrooms we must clearly recognize the specific values assigned to this variable in the culture of which we are a part. The propositional function that "Mr. Kline is a responsible citizen" says one thing if it comes from the pen of Adolf Hitler and a very different thing if its author is Franklin D. Roosevelt. We live in a democracy and the schoolrooms of America are symbols of our faith in the power and significance of free public education. They are great laboratories of democracy where our potential citizens should actively participate in experiences essential to the democratic process. Nor is the responsibility for providing such experiences limited to the teachers in any single field for the obligation to build ideals of democracy into the characters of our young people is just as binding on the teacher of mathematics as it is on the teacher of social studies or English. Our responsibility does not end with the development of skillful computers although in no sense would I belittle the importance of the skills and operations of mathematics. In fact, their importance in dealing

effectively with the problems of our physical environment cannot be exaggerated but they will never serve any such useful purpose unless associated with them are conceptual understandings and insights of even more importance because they are essential to the selection of the needed operations. Responsible citizenship in a democracy cannot be expected from young men and women who are operators only, for they are no more adequately equipped to apply the methods and procedures of mathematics to the solution of the problems with which they are daily confronted than is the sluggish elephant who on a circus platform demonstrates his ability to add two and three. They are like the snakes of Noah's ark who, in response to the command of the Lord to go out and multiply, replied that they were unable to obey because there were "adders."

In the schools of Germany between 1933 and 1939 the operational techniques of mathematics were undoubtedly learned as effectively as anywhere else in the world, as the sons and daughters of passive parents were being prepared for "responsible citizenship" in a Nazi culture. However, during these fateful years pregnant with disaster for the peoples of the earth, the teachers of the Third German Reich carried into their classrooms their warped and distorted ideas and through a process called "education" led their obedient students to accept the vicious proposition that they belonged to a superior race and were destined to rule the world. The ability to examine the quality of evidence supporting this proposition, to analyze the assumptions on which it is based and to use related understandings associated with the nature of proof was not developed through their study of mathematics nor would such outcomes be tolerated in a Nazi society. The skills and operations of mathematics may be its only contribution to responsible citizenship in an antidemocratic culture but much more is expected from the study of mathematics in the classrooms of a democracy.

The Harvard report, in discussing the problems associated with "General Education in a Free Society" recognizes the tool value of mathematics and then proceeds as follows:

> Beyond this, however, mathematics has an important intrinsic role in general education. The ability to analyze a concrete situation into its elements, to synthesize components into a related whole or to isolate and select relevant factors, defining them rigorously, meanwhile discarding the irrelevant and the ability to combine these factors often in novel ways so as to reach a solution, all are important features of mathematical procedure.

A similar emphasis is made in the report of the Joint Commission while an entire chapter of "Mathematics in General Education" considers the importance of providing the students with experience in selecting, securing, organizing and interpreting the data pertinent to a problem of interest and concern to them.

This is a sound and healthy emphasis for inherent in it is a recognition of the fact that responsible citizenship in a democracy calls for a functional understanding of the problem-solving process which is at the very heart of democratic procedure. Was the fine of three and one-half million dollars imposed upon the United Mine Workers through the whim of a dictator, or did Judge Goldsborough find that the data pertinent to this significant controversy called for such a penalty? How will the Supreme Court of the United States determine whether or not the decision of the judge is or is not constitutional? Will their decision be based upon their regard for Mr. Lewis or will it reflect their own, honest interpretation of the data relevant to the problem? By what process will a decision be reached concerning the problem of tax reduction? Will members of the Congress listen to the voice of Harry Truman or to the voice of Robert Taft? Consistent with our democratic society, they will listen to each of these gentlemen as well as to many others and their final decision will be reached on the basis of all available data. As a realist,

you will probably say that I am altogether too naive and that a major factor in the final decision will be the political considerations involved. You are undoubtedly correct but when you say that you are only saying that political considerations constitute an important part of the relevant data, as indeed they do. During a period of 14 years thoughtful voters have accumulated data concerning the effectiveness of New Deal policies and regardless of political desire one must admit that the interpretation of these data by responsible citizens of the United States has resulted in a Republican Congress.

The problem-solving process is indeed at the very heart of democratic procedure and mathematics can be so taught as to develop students who are skillful and effective in the use of this process. Such desirable results, however, will not be achieved by teachers whose weary voices echo throughout the mathematics classrooms of America assigning for homework the next 20 problems on page 146. If responsible citizenship calls for the selection, the securing, the organization and interpretation of data relevant to a particular problem then it is our obligation to provide our students with that kind of experience and I would vigorously support the recommendation of the Joint Commission that "in the future definite provision must be made for training of this kind." How tall is that flagpole in front of the school? What is the area of that lake? Is our classroom sufficiently large to meet the optimum conditions of 20 square feet of floor space per pupil? Is the height of the ceiling one-half the width of the room as required by the Ohio building code? Shouldn't my allowance be increased? Shall I put my money in the savings bank or shall I purchase government bonds? Let students have the opportunity to determine the data needed in the solution of problems of this sort which is an exercise in relational thinking far more fruitful than wondering how y behaves when x is doubled in the formula $y = 8x$. Make it possible for them to gather these data and guide them as they use the principles of mathematics in their effort to determine the conclusions to which the data lead.

The actual solution of these problems may be important to the young people who solve them but still more important is a steady and persistent emphasis on the process by which they are solved. As teachers in the schools of a democracy we have an obligation to build into the intellectual equipment of our students the ability to use this process in dealing with the wider problems of life and as teachers of mathematics we have available an instrument through which we can make a most effective contribution to this highly desirable outcome.

A further point which should be recognized in our consideration of mathematics for responsible citizenship is the increasingly extensive use of mathematical concepts for the expression of significant ideas. The extent of this practice warrants the assertion that to be ignorant of the language of mathematics is to be unable to read current literature with intelligence. In October 1943, when the so-called "Big Three" met at Moscow, a full page picture of Joseph Stalin appeared on the cover of "Newsweek" and this distinguished gentleman was referred to as "The X in the International Equation as the Moscow Conference Opens." The meaning wrapped up in this concise and suggestive statement is forever hidden from the man whose intellectual development does not include a conceptual understanding of the equation.

What is understood by a reader when he is informed through the columns of his daily paper that:

Congress is preparing to do violence to our inch.

The new day will have 20 hours and each hour will have 100 minutes.

The two-millionths of an inch is well within the limits of tolerance.

By order of Dr. Wilhelm Frick, Minister of the Interior, the German Land Survey Department will adopt a circle of 400 degrees?

How can man tamper with these fundamental measures, so essential to society, and what is the effect of such statements on a bewildered reader? Will he agree with the lady who wrote to a member of the World Calendar Association, accusing him of being a heretic, and saying, "I want the calendar to stay in its present form just as God made it," or will his conceptual understandings be such as to recognize that all measurements have no real existence in any absolute sense and are valid only because of agreement?

A recent advertisement of the Comptometer Company appearing in many magazines pictures a googol, accompanied by a statement that the number of rain drops falling on Chicago in a century is less than a googol and states that, "When one is dealing with such figures, accuracy is not to be expected." What does the author means by this statement? Is accuracy dependent on the size of the numbers used? When the Federal Government reports that American casualties number 1,049,104 is that an accurate number or is it "too large" to be accurate? In a recent publication the population of the United States in 1940 is given as 131,669,275, while that of Russia for the same year is 165,847,000. Should a reader consider that each of these numbers is accurate? Is either of them accurate or are they both only estimates? As of June 1, 1944, the Bureau of Census at Washington "estimated" the population at 138,100,874 persons. Is this number as accurate as it appears to be or is this appearance of accuracy misleading? Reliable interpretation of these and similar statements which are constantly appearing in the daily press requires a familiarity with the language of mathematics which investigation reveals is sadly lacking among our "educated" citizens. They may be effective and dependable operators, but they fail to recognize that the results of their operations are no more reliable than the accuracy of the numbers they feed into them.

There recently appeared in a national magazine of wide circulation an interesting article in which the ideas of a number of important political figures were analyzed and contrasted. Some of these gentlemen were natives of Ohio and in the course of a brilliant discussion in which significant and deep-seated differences were clearly defined, the author pointed out that "these differences could undoubtedly be justified by the non-Euclidean character of Ohio politics." What does he mean by this strange and unfamiliar expression? What is suggested by the term "non-Euclidean" and what does it mean when used in this context? The author has drawn upon his mathematical understandings to enrich the quality of his writing but unfortunately this very suggestive statement will have little, if any, meaning for 99.44% of his readers. Official records reveal that the number of students who graduate from high school with no knowledge of Euclid is steadily increasing and here is a man, writing for the intellectual consumption of laymen, who glibly assumes that readers understand the significance of such a term as "non-Euclidean." Mathematical concepts are indeed widely used in the discussion of current problems and responsible citizenship cannot be expected from men and women who are unable to read these discussions with insight and understanding.

One of the fundamental assumptions underlying the democratic process is faith in the judgment of the common men and one of the major responsibilities of our system of public education is to improve the quality of that judgment. The powerful forces which released the spirit of man from crippling bondage and placed in his hands the responsibility for making decisions of far-reaching significance also released a variety of pressures designed to affect the quality of those decisions. In this new world of intellectual and spiritual freedom which man has created there are still potential dictators who would tell him what to do and from every direction he is besieged by propaganda which would

distort his judgment. Against these dictatorial forces he needs the protection of an intellectual integrity which is essential in dealing effectively with the varied and complex problems faced by every citizen of a democracy. Governments do indeed derive their just powers from the consent of the governed and in that assumption, so fundamental to our democratic way of life, there is a further assumption that our citizens know how to evaluate the forces which operate in a free society, and that their final judgments are the result of sound methods of thought. Does it stretch the imagination to believe that mathematics is an instrument which can be used to improve the quality of those judgments? Is there no potential relation between the methods of thought which lead us to "consent" to the proposition that "the sum of the interior angles of a triangle is 180°" and the methods of thought which lead us to the conclusion that "Federal taxes should be reduced"?

Perhaps you believe that I am placing too great a burden on mathematics and that the only contribution which "the queen of the sciences" can make to responsible citizenship is found in the language by which we can deal with the quantitative and spatial relationships of life. If that is your position, then I would ask you to listen to Professor J. W. Young who stated 22 years ago that the chief objective of mathematical study is "to make the pupil think" and "if mathematical teaching fails to do this, it fails altogether." Are these the exaggerated statements of a skewed and distorted mind? Are they worthy of your attention or do you dismiss them with a shrug of your weary intellect, as you recall that even now, without this added emphasis, your students have exasperating difficulty in getting the right answer? Would you also dismiss the statement of Earl Raymond Hedrick that the processes of mathematics are "of constant occurrence and of wide application in life activities"? Would you deny that the process by which conclusions are established permeates the fabric of our democratic culture and that according to Professor Hedrick this process is "taught more consistently in mathematics than in any other subject"?

These men are distinguished mathematicians who command the respect of the academic world and their position concerning the potential educational values in the study of mathematics cannot be ignored. But because they are mathematicians there are those who would say that to accept their position on this particular issue is comparable to accepting the position of a Nazi judge as to whether or not Herman Goering is guilty or innocent of the crimes for which he was tried at Nuremburg. Max Wertheimer, however, is not a mathematician and in his book on *Productive Thinking*, after examining a number of geometric examples, he makes the following significant statement: "In our geometric tasks it is possible to proceed in a transparent, crystal-clear way. This was one of the reasons why educators again and again urged the study of geometry, which allows the development of mental abilities in an atmosphere of clear, transparent consistency and which may then help toward similarly straight attitudes in thinking about more complex and less concise matters." The Harvard Report to which reference has already been made places a heavy emphasis on logical and relational thinking as important outcomes in a program of general education. Not one of the distinguished gentlemen who signed that report is a mathematician but included in it is the statement that while "mathematics is by no means the only road to an appreciation of abstraction and logical structure, these ends may be approached most readily through mathematics, particularly with the young. No better example of an abstract logical system for use with adolescents than demonstrative geometry has yet been discovered." Consistent with these ideas is the statement in the report of the Joint Commission that "mathematical methods

set up an ideal, a norm, a pattern, which though not attainable in all the activities necessary and proper to man, serve as a relentless spur to improvement. . . . "

Are we using mathematics for the achievement of these important outcomes? Do we develop in our students an appreciation of the power and importance of logical and relational thinking? Are they really familiar with the character of mathematical methods and procedures and are these understandings generalized for use in wider fields of thought? If we could with honesty answer these questions affirmatively not a single voice would be raised in protest against the time spent in your classroom and in mine but from every college campus, from every military center and from the great laboratories of life there comes increasing testimony that we have failed to realize these larger purposes. Our earnest and persistent concern for operational efficiency has caused us to lose sight of the broader educational values which also call for steady and continued emphasis. Reliable conclusions are important but of equal importance is a penetrating insight into the nature of the process by which these conslusions are determined.

Let no student in your classroom or mine develop the idea that mathematics is nothing more than a tool for getting an answer to an isolated problem. Let us rather lead him to recognize that mathematics provides a powerful method for dealing with the problems of life and that its principles and procedures are available in wide fields of human activity. Let us be faithful to the implications of the following paragraph from the report of the Joint Commission:

> . . . but there should be conscious experience with both inductive and deductive reasoning. The character and requisites of these two procedures should be so clearly grasped that appropriate behavior on the part of the pupil is brought about in the direction both of understanding and of making applications. . . . He should seek to discover and remove ambiguity in the use of terms. He should understand the relations between assumptions and conclusions and he should grow in the ability to judge the validity of reasoning which purports to establish proof. Proper attention must be given to the generalizing of these behaviors and understandings. We may then hope that pupils will apply them to situations arising in many different fields of thought.

A similar emphasis is found in the proposals of "Mathematics in General Education" and in each of these major reports there is explicit recognition of the fact that if mathematics is to make its greatest contribution to responsible citizenship its principles and procedures must be generalized for wider application. The Harvard Report recognizes the potential contribution of mathematics to the development of effective thinking and then states that this ability "will not be fostered as it should be even in mathematics unless the logical movements which find their purest form in theorems and equations are expressly given wider use. . . . The projection of the structure of geometry into areas of more immediate and often of more practical interest to the student should be taught explicitly. It is only in this way that there can be accomplished the transfer of mathematical values to other spheres of human interest which is a primary concern of general education. . . . "

There is accumulating evidence that teachers of mathematics who are faithful to these ideas are providing experiences in their classrooms which do promote "the transfer of mathematical values to other spheres of human interest" and which do develop "straight attitudes in thinking about more complex and less concise matters." These larger values are reflected in the behavior of the students and their importance is well illustrated by the comment of a father concerning his son when he said, "His method of dealing with the problems he faces is convincing evidence of the value of his mathematical training." This satisfied father did not mention the operational skills and spatial understandings which his son had acquired in completing two years of high school mathematics. He was undoubtedly impressed by

the value of these skills and understandings but he was more impressed by the effective manner in which his son applied the methods and procedures of mathematics to the wider problems of life. Too many of our students regard mathematics as a heterogeneous group of unrelated operations and skills. They have developed no generalized understandings and echoing through our classrooms is the often repeated question of their worried and anxious parents, "How is Willie doing in mathematics?" To ask such a question is to reflect Willie's limitations for observing parents should be able to answer their own question since mathematical understandings essential to responsible citizenship are reflected in behavior. The real question about which they should be concerned is not "How is Willie doing in mathematics?" but rather "How is mathematics doing in Willie?" and the best way to find a valid answer to that important question is to observe Willie's behavior as he grapples with the problems of life.

Upon the schools of America is placed the important obligation of developing young men and women who are equipped to meet the responsibilities of citizenship in our democratic society. A mathematical program designed to contribute to the achievement of this desirable purpose would place steady and persistent emphasis on the problem-solving process. Students would be encouraged to define problems of interest and concern to them, to select the relevant data, to gather and organize these data and to draw valid conclusions. The principles of mathematics would be the outgrowth of experience rather than the basis for it while operational skills would serve a recognized and useful purpose. The students would have experience in clarifying the definition of essential terms, of detecting and making explicit needed assumptions and in examining the validity of conclusions. Throughout this entire process there would be continuing experience in establishing conclusions by both inductive and deductive reasoning and the abilities thus developed would be generalized for use in wider fields of thought.

Nor is such a program designed especially for those students whose promise of achievement in the field of mathematics is particularly inviting. They alone do not determine the quality of our social fabric, for the responsibilities of citizenship are not limited to those whom inheritance has favored with an I.Q. of more than 110. They come to the intellectually rich and poor alike and there is evidence that a mathematical program so designed can be instrumental for improving the thinking of students at all I.Q. levels. My own experience verifies this conclusion and it is my humble judgment that through such means we can help all the young people in our secondary schools to become more resistant to the pressure of propaganda, more critical in their own thinking, and more intelligent in meeting the responsibilities of citizenship in our democratic culture.

Attention National Council Members!

Due to the increased cost of printing THE MATHEMATICS TEACHER, the subscription price of the magazine (i.e. annual membership dues) was raised by the Board of Directors of the Council at the recent Annual Meeting at Atlantic City to $3.00 per year; single copies 40¢ postpaid. This increase in price takes effect October 1, 1947. This means that all members whose subscriptions expire in May should send $3.00 for their renewals and of course all new members should remit $3.00 also. Mr. W. F. Crawl of Atlanta, Ga. renewed his subscription at $3 and said "Let's keep THE MATHEMATICS TEACHER going if a year's subscription goes to $6. It is worth every cent of it! I would not be without it if you were to treble the price." Of course we hope we can keep the price at $3, at most.—EDITOR

On-the-Job Training of Teachers

By ROLLAND R. SMITH
Coordinator of Mathematics in the Public Schools, Springfield, Massachusetts

"TO IMPROVE teaching is to improve the schools; to improve the schools is to strengthen the next generation; to strengthen the next generation is a social duty of the first magnitude."

One hundred twenty five thousand substandard teachers! So goes one report. Maybe the number is not right. Possibly there are more. It could be that there are fewer. At any rate we know that we have a difficult problem in this respect and that the problem is growing steadily more acute. Do you have some of them in your system? If so, do not *blame* them. Help them. Even if you do not have any substandard teachers, you still need to give all the educational help you can. The problem of in-service training of teachers is not something new made necessary by the present emergency. Excellence in teaching implies continuous growth. Even the best of teachers, or should I say especially the best teachers, are never satisfied. They want to grow. You would not long keep the services of a medical man if he were not attempting to keep up to date even though he were at the head of his class in medical school. Neither can we keep an educational system worth the name if our teachers do not continue to learn.

Often, when we think of in-service training of teachers, we think of courses taken in some school of education. These courses, if they are good, are invaluable as a background. They help to keep the teacher up to date. They give new ideas with which to experiment. They improve the teacher's working knowledge of the psychology of his subject. They can point out dangerous situations to avoid. They can discuss desirable outcomes and they can help with many of the teacher's individual problems if he is alert enough to see clearly what his problems are. I am not saying anything new when I recommend such courses highly and urge all teachers no matter how much experience they may have had or how many degrees they have to take advantage of them.

However, something further is needed. Often these courses are not taken by the teachers who need them most. Many times I have heard Dr. Reeve say, "You are the cream of the teaching profession. Otherwise you would not be here." And then too, I am thinking of the thousand and one problems that arise in the classroom day after day and cannot wait for an answer until the teacher goes to summer school. I am thinking of the questions that arise in connection with the interpretation of courses of study, in connection with the most efficient use of the adopted textbooks, what to do with these few pupils in the class with high intelligence, and what to do with all these backward pupils—all questions that in the last analysis must be answered by the details of procedure of the classroom teacher. Many of the problems which at first sight seem to be purely administrative or curriculum problems must finally find their solution in effective classroom practice. For these reasons I am interested in on-the-job training of teachers. And I mean just that—training of teachers during the time they are on the job.

I feel strongly that every school system of any size should have a supervisor of mathematics whose main object is not to find out which teachers are good and which are not so good, not to visit teachers to be able to tell them what their faults are, nor yet to sit in his office and send bulletins requiring extra work of the teacher so that he can make statistical reports to his superiors and justify his existence, but

whose main purpose is to help teachers grow professionally. Such a man (or woman) would have to be good. First of all he would have to be a superior teacher because he would give a great deal of his time actually teaching in classrooms throughout the system to learn at first hand what the problems are and to help teachers by giving demonstrations. He would have to be up to date and creative in his approach to curriculum problems for at this time particularly our attitude toward the kinds of mathematics to teach various groups is changing rapidly. Teachers with five of six classes a day need continuous help. He would have to be democratic and have a good personality and strong leadership qualities because teachers resent dictatorial methods and cannot learn unless they are made co-workers in the building up of mathematics courses and ways of handling them.

Such a man (or woman) may be hard to find but he exists and can be found if school systems want him badly enough. What usually happens to such a superior teacher at present? He cannot be recognized as a superior teacher because salary schedules allow very little for recognition of this kind. In order to advance he must become an adminstrator. The system loses a good teacher, it often gets a poor administrator, and no one is happy—least of all the superior teacher. I propose a different method of advancement. Let him be advanced to the kind of creative and teaching supervisor that I have been describing. Then he can continue to be a teacher. He can continue to have the feet-on-the-ground attitude of the classroom teacher so often forgotten by the teacher who becomes an adminstrator. Most of all, he is ever present in the system to give help where help is needed. His job will be to help teachers grow professionally.

I have not forgotten that I am talking about on-the-job training of teachers even though I have given some time to the supervisor. I have done this because I believe he is the key to the situation. He can get himself involved in many things that only remotely help the teacher, he can be an autocrat who hands down orders to obsequious or resentful teachers, or he can point everything he does—choosing textbooks, developing courses of study, testing pupils, even making reports—toward his most important objective; namely helping teachers to become better teachers.

I should like to divide this supervisor's job into two main parts, (1)developing courses of study, and (2) giving classroom demonstrations, and talk briefly about these two parts. How can the supervisor deal with curriculum problems such as the development of courses of study so that teachers learn as the courses are developed? First of all he will not try to write a course of study to make a name for himself. Teachers can see through such a scheme very readily. Courses of study written to send out to another city with the idea of "what a good boy am I" never helped teachers or pupils much. Most often they lay in the teachers' closets and gathered dust. Fortunately the day of the voluminous textbook type of course of study is on the wane.

Teachers can be helped by being called together in small discussion groups to study modern trends in mathematical education. They need to know that a revolution has taken place in the last twenty years in our thinking about the teaching of mathematics. An earlier psychology taught us to break every teaching topic into as small bits as possible. This topic was broken into 1492 fragments and that topic was broken into 1776 parts and each part was taught in isolation. Now we have turned completely in the opposite direction and find that teaching is far more effective when large general ideas are given first. Groups should be small enough so that the meaning of this change can be discussed thoroughly and translated into actual classroom teaching. Teachers, too, need leadership in the study of the findings of such committees as our Postwar Commission. If left to themselves most

teachers would read the reports and say, "Some good ideas. I wonder when our system will try to put some of them into effect." They do not get enough detail to encourage them to experiment on their own account. Group thinking with strong leadership will help them not only to a well developed philosophy of teaching but also to changes in their classroom practice.

When a philosophy is decided upon and the topics in any one course are agreed to in general, the next step is a choice of textbook. And again this can lead to teacher growth if it is handled democratically. Please note that I have put choosing a textbook before the writing of a course of study but after the philosophy and general outline. I have given the textbook writers the benefit of the doubt and assumed that they have been able to deal with all the details of a good course better than most groups of teachers. After this the course of study will take on the complexion of a teachers' manual, and that is as it should be. The textbook is the course of study but it has to be fitted to local needs.

Let us take one or two examples to see what can be done now. Let us assume that we have a good seventh grade textbook and that all seventh grade pupils, except the very lowest group, should cover the same topics in general. Teachers want to know how slow groups can cover the same topics as the faster groups. They need to learn that it can be done if the slow group covers only the essential relationships while the faster groups go into the topic more deeply but not necessarily more thoroughly. Groups of teachers should get together and study the book page by page to discover the psychology of the book, to determine where it needs to be supplemented, to see what can be omitted with the slower groups. When this has been done, the suggestions should be gathered together and mimeographed as a teachers' manual. This is not the end of the matter, however. The textbook is not studied once for all. It is a continuous process as teachers get more experience with it and further study brings forth new suggestions. Pretty soon a better textbook is found and the process begins all over again, albeit we hope on a higher level. With proper leadership teachers are showing continual growth.

There are many other curriculum problems to be handled in a similar manner. Can we come to any conclusion concerning what is to be done with that lower tenth in the seventh and eighth grades that do not seem to be able to learn much of anything? What has been written about them? Is ungraded classes in the junior high school the answer? If we can get little help from the literature, can five or ten teachers who have common problems sit down together and learn something from a pooling of their intelligence and experience? Teachers cannot help but grow under this kind of procedure.

What do we mean by a second track from the ninth grade on? How shall we determine who shall study what? Do we change our methods of teaching with the lower groups? Should we teach applications of mathematics and get the mathematics incidentally or should we teach the fundamentals of mathematics first? What is meant by the Postwar Commission when it says that in geometry large ideas are more important than details? Does this mean that we should not be thorough but only skim over the high spots? Can general ideas be developed without first teaching thoroughly the details which compose them? Questions like these can be discussed in small groups, not in general but with particular classes and pupils in mind. The discussions should be planned to produce effective changes in classroom procedure.

So much for the training of teachers in connection with curriculum making and connected activities. What is learned can be more easily put into practice than the ideas culled from courses in education because it has been learned directly in connection with the work of the classroom.

Still many teachers require something more specific. You can talk individually and collectively to teachers about keeping pupils alert and active all the time and give suggestions as to how it can be done. You can suggest that by proper techniques you can learn what pupils have already mastered and so save much time, energy, and boredom by not teaching what they already know. You can talk to them about methods of covering courses of study in poor as well as in good classes. You can discuss the necessity of the interplay of the concrete and abstract to make meanings clear. You can even go so far as to discuss specific problems with as much clarity as you can. For example, you can discuss the procedure in giving a general picture of meaning and practice with decimals before going into greater detail. And yet teachers will still tell you that they get the point of view better when they see things done in the classroom than when they are talked about. And when it comes to making clear to teachers what you mean by watching every reaction of the pupils and reacting accordingly there seems to be no other way than to demonstrate your meaning.

Teaching is an art as well as a science and artistry can better be developed by seeing an artist work than by hearing him talk. Even then you cannot be sure of results. Provision must be made for practice by the observer under proper guidance. Perhaps we can understand some of the attributes a good teacher must have if we compare him with a good salesman. The salesman prepares his prospective buyer but he must know precisely the time to bring out his order book and get the buyer's signature on the dotted line. If he goes beyond that time, he may lose the interest of the buyer and the sale as well. The salesman may lead the buyer up to a certain point and then decide on the spur of the moment that it is better to tell a good story and come back another day. He may find that he has made a false start and have to backtrack gracefully in order to make a different approach. He may have a lengthy demonstration prepared only to find that the buyer is way ahead of him and does not care to go along so slowly. And then, after he has made the sale, if he is the kind of salesman who always has more things to sell, he must make sure that the buyer knows how to use the product he has sold. He must have a well defined service program. At the same time, he must be always going ahead with the new products he has to sell.

To do all this the saleman must, of course know his product. But he must also know the needs of the buyer and the kind of man he is. The salesman has to watch every reaction of the buyer, his posture, the gleam, the twinkle, or the apathy in his eyes, the tapping of his foot, the glances toward the clock. To be a good salesman he must be almost able to read the mind of the buyer, to foster moods in him, and then time his actions to the proper mood.

Just so it is in the teaching of mathematics. An adequate grasp of subject matter and general methods is assumed. Then the pupils must be prepared for each topic, usually by means of a concrete approach. But it is easy to carry the approach beyond the pupils' needs. If this is done, they lose interest, they forget what they are doing, the right moment for bringing in the more abstract parts of the topic is lost. On the other hand, the concrete may not be carried on long enough and then the abstract is something mechanical without meaning. As in the case of the salesman, the only possible solution of the problem is to watch and evaluate every reaction of the pupils. As soon as you think they are ready, even before, make a jump into the abstract as a challenge to the better pupils. If you find that the class is not quite ready you can always backtrack and begin your development again. If you watch carefully you may find that your pupils are ready sooner than you think.

And then in reservicing a topic, you must catch your pupils just before they

forget, not always an easy task. And in going on to a new topic you have to be watchful. It is so easy to stay on one topic far too long. We try for perfection when we know we cannot get it an any given time. The ideal of perfection can be responsible for much loss of time and morale. The time to begin a new topic is not when the pupils have reached perfection in the old topic, but when they have sufficient understanding to begin organized drill. Progressive understanding comes not from staying on a topic until temporary perfection is reached but by returning to it again and again.

Now it is very difficult to talk to teachers about these subtle details of good teaching so that they know what you mean and can go and do likewise. But a superior teacher can go into a classroom and demonstrate what he means. The teacher is not expected to imitate but he should be able to get the point of view.

Demonstration of good teaching, I believe, is the heart of the whole matter of on-the-job training of teachers. Discussions should usually precede the demonstrations. Teachers need to learn what to look for in order to become analytic in their observation and criticism. Then there should be an opportunity afterward for the teachers to discuss the demonstrations. Various demonstrations throughout the system should help in giving insight into the finer points of teaching which I have just discussed as well as in throwing light upon some of the larger problems which arise in connection with the curriculum. Incidentally the supervisor as well as the teachers will learn a good deal.

When problems arise anywhere in the system, the good supervisor will try to become familiar with them by going into the classroom and teaching, not because he knows everything but because he has the experimental approach. He cannot always be right but he can always be willing to try. Often it will be necessary for the supervisor to take a class for only a day. Sometimes he will have to take the same class for a week or even two weeks in order to watch progress or the lack of it, to be able to see the results of his teaching or to keep on trying when first attempts fail. Always as many teachers as possible (and the principal as well) should witness the demonstrations and study them analytically.

I should like to conclude with a brief description of my idea of the handling of a particular problem. Suppose, for example, that the supervisor has visited a certain junior high school and finds that a teacher is having difficulty with a ninth grade general mathematics class. The supervisor then discusses the problem with the teacher. Next he visits other junior high schools and finds that most of the teachers of this subject are having the same difficulty. The problem is this. The children in these classes do not know much arithmetic. The teachers feel the problem keenly and have a tendency to keep at arithmetic constantly. They have no time to bring in other topics like geometry, scale drawing, indirect measurement, and a little algebra. They are worried about it. To be sure a committee of teachers working the year before had suggested that half the time given to arithmetic would probably give as good results as a full time job. There comes a time of diminishing returns. Then the other half of the time could be given to many of the topics suggested by the Postwar Commission in its Second Report. A split period technique had been suggested. The poorest classes were to spend not more than half a period on arithmetic and the better classes less time in inverse proportion to their ability. But habit and tradition are strong and the teachers were still giving most of their time to arithmetic.

In order to become personally acquainted with the problem the supervisor will now take the first teacher's class for a period of perhaps four consecutive days. At the beginning of the first day he gives a ten minute test in arithmetic consisting of addition, subtraction, and division of

whole numbers, division of fractions, and a problem in scale drawing. He finds scattered errors in the first exercises and rather general difficulty with the last two. So he spends about fifteen minutes discussing these two exercises and the principles involved. He now has thirty minutes left to go on to geometry. The second day his plan is much the same. The teacher suggests that on the third day the supervisor discuss percentage which has caused a good deal of difficulty. The supervisor does this and finds a redevelopment necessary. In fact he spends the whole period on it and knows even then that he has not accomplished much with percentage. And yet on the fourth day, after a few questions on percentage, he evens things up by spending most of the hour on geometry. He found that he could not keep to the letter of the split period technique but he had kept to the spirit of it.

After these four days in the one class, the supervisor goes to other junior high schools and takes charge of similar classes. Since he finds much the same situation in these classes, he spends only one or two days in each class. He is now ready to call the group together for a discussion of common problems.

I believe that several things have been accomplished. The supervisor has gained first hand experience with the problem and is not asking the teachers to attempt something he has not tried himself. The teachers have had the inspiration of new insight into their problem. They have seen how the pupils react to the procedure suggested by the course of study. They are ready to sit down around the discussion table and in the give and take of democratic procedure come a little closer to grips with their problem.

I believe I am right in calling all this *on-the-job training of teachers*. It takes place entirely within the system, in large part during the time when classes are scheduled. There is a definite program for studying trends and the findings of research committees. Teachers study these things in the light of their actual classroom experiences. And finally they have a chance to see the interpretation of all these things in classroom demonstrations. It is a continuous procedure and should result in a raising of the level of the teaching of mathematics.

The Decimal Point*

I am the decimal point, rolling along,
Gee, but you've had your fun getting me wrong,
Putting me here, and putting me there
Any old place, any old where—
But I really am important,
It's the truth, it's act-ch'll
I make your answer satisfact-ch'll
I am the decimal point, rolling along
Please, oh, please put me, where I belong.

* You have had such clever songs and poems in THE MATHEMATICS TEACHER in the past that I thought you might like to publish this parody on the tune of "Zip a Dee Doo Dah" from the recent movie, SONG OF THE SOUTH. Miss Julia Diggins, a mathematics teacher at Paul Junior High School, put on a mathematics assembly, and wrote this for a skit in which a little black(ened) decimal point danced around among a lot of digits on the stage to show how he changed the value of the number by moving around. It was very effective.—Veryl Schult

The Place of Multisensory Aids in the Teacher Training Program

By HOWARD F. FEHR
State Teachers College, Montclair, N. J.

WHEN Perry made his now famous address to the British Society in 1901, he advocated a change in teaching, away from the pure abstract approach, and in its place, the use of scaled drawings, graphs, models, physical apparatus, and practical applications, so that pupils would achieve meaning or "mental tools as easy to use as their legs or arms; enabling them to go on with their education (developing of their souls and brains) throughout their lives, utilizing for this purpose all their experience."

Professor Forsyth in rebuttal defended pure abstract mathematics, using the following illustration: Legend has said that there is in Labrador, for in the interior, a beautiful waterfall far finer than Niagara. Many men set out to find this fall but most gave up along the way and returned home. The way was beset by dense forests, deep ravines, bogs, quicksands, and high walled mountains. Only nine men overcame all obstacles and beheld the glorious sight. So it is with pure mathematics. It is difficult, abstract, complex, and beset with ever increasing and almost unsurmountable difficulties. But to those nine men who wade through the bog and scale the high walls, there is the reward of a glorious waterfall,—the intrinsic beauty of a complete understanding of number and space.

Perry replied that despite the pure mathematicians, the engineer would build a bridge over the ravine, a railroad over the bog, steps up the wall, and cables across the falls. The beautiful scene will then be thrown open to hundreds of thousands of white men, and moreover the waterfall would be used by the engineer to create power to give more wonders and comfort to the human race. And despite the pure mathematician he proposed to do the same thing for mathematics.

Any discussion of the use of particular media in the training of teachers must be predicated on the primary purposes and desired outcomes of such training. If we know what ultimate final product we expect, then we can plan the machinery for producing that product. It may be safe for our purposes, to answer that we expect a teacher of high school mathematics to have studied, and to have an understanding of, collegiate mathematics to, and even beyond the calculus; to know a fairly large number of applications of mathematics; to be thoroughly familiar with all high school mathematics; and to know various methods of presenting his subject to high school pupils so as to produce correct and meaningful mathematical concepts. We assume that a teacher improves in respect to all these various requisites as he continues to teach.

Mathematics, as a pure logical science consists of abstractions, of very high order, concerning number and space. Mathematics as taught in the high school and the colleges, also consists of abstractions, but those in which the logic is of lesser importance, and intuition or common sense is more prominent. These abstractions are made from concrete physical objects or from statements concerning these objects. I believe that today most mathematicians would agree that mathematical study had its origin in concrete physical objects of our physical world and that much mathematical theory is accepted or discarded, according as it does or does not give satisfactory descriptions of physical phenomena.

If this is so, it would seem reasonable that the teachers of mathematics, in high schools and colleges, should not hesitate to use concrete physical objects to aid in the establishment of mathematical con-

cepts that originally arose as abstractions from the study of these objects. Plane geometry as taught in high school is an example of this process. Perhaps one out of a million pupils studying plane geometry creates for himself a logical development of abstract pure geometry of ideal points, lines, planes, and circles. Most pupils are proving facts about a concrete figure. Every plane geometric figure we have seen has been a physical concrete model either on the blackboard or printed page, or made of some physical material. The fact is we cannot see an ideal triangle. We can define it, prove properties concerning its parts, and talk about it. And to prove these relations in the abstract, without a figure, soon becomes an almost impossible task for all of us. Try proving a difficult original without a model. Thus in geometry we prove relations by the use of physical figures, and by the use of relations abstracted previously from other physical figures.

Number also begins with physical objects as we learn to count them, and then add, subtract, multiply, and divide them. The abacus merely substitutes a model set of physical objects for carrying on these operations. However number and its generalizations in algebra are increasingly more abstract and more difficult of comprehension, than the similar development of geometry. This may be one reason for the apparent failure of algebraic study to contribute to the educative process. Pupils never did learn algebra, because they never did abstract, but merely repeated meaningless mechanical operations on letters. Hence any concrete device that can help pupils see the abstractions in algebra, and thus give meaning to the operations, should be used and should be welcomed by the teacher.

The use of multi-sensory aids in the teaching of pupils with average intelligence or lower is unquestionably necessary. These pupils must learn facts, see these facts applied in practical situations and then make corresponding applications in similar situations. This does not mean that pupils of superior ability do not need to use sensory aids. It does mean, however, that their use of sensory aids will be made in a different manner and for different purposes from those in the case of lower ability pupils. Since in teacher colleges and university schools of education we assume only superior students, our discussion will be limited to the place of sensory aids in teaching these students.

The future teachers of mathematics will make use of sensory aids in their own study of college mathematics. They must also learn the proper use of such aids in the teaching of high school mathematics. While we are advocating a five-year training program, we must be realistic enough to recognize that for quite a few years to come, we can only expect a required four-year program, which is already overcrowded with required subject matter. We can expect that many teachers will continue their education while in service, and for these teachers we can create courses in audio-visual education. But to some extent, we must provide an adequate coverage of methods of sensory education during the four-year program. To accomplish this will mean the curtailment or elimination of work heretofore required, and this will not be easy. One proposal to accomplish this will be made at the end of this article.

Perhaps the simplest mechanical aid is the graph of a function. Functional notation, independent variable, and dependent variable are extremely abstract concepts for the beginner in the study of analysis. So we resort to a geometrical and mechanical device of changing numbers to geometrical lengths, and we study the variations in the geometrical figures, in order to clarify our knowledge of functional variations. No one objects to this today, indeed, almost all teachers insist that it be an integral part of our instruction. Yet, as late as 1893, the 4th edition of Charles Smith's "Treatise on Algebra" contained not a single graph, number scale, or geometrical

diagram in its entire volume. Hall and Knight's "Higher Algebra" likewise contained no graphs, until it was revised by Sevenoak in a third edition published in 1913. Then a last chapter was appended on the graphs of functions.

The graph, however, is not the final product. It should serve merely as an aid in establishing meaning to such concepts as the number of and reduction of roots, maximum and minimum values, inflection points, etc., all of which are defined analytically in terms of variables. The engineer today frequently, and perhaps in the majority of cases, solves his equations by the use of graphs and their intersections. He does this because he understands and sees what he is doing. He has long ago forgotten Horner's or Newton's method, or what is more likely, never understood the algebraic abstract concepts at the base of these methods. Today we use the graph to explain the concepts behind these methods to the end that the method, and not the graph becomes the desired goal.

But the graph is not the only sensory aid. Whenever the use of some concrete physical model will establish an analytic relation in the same or less time, and with clearer insight, than could be obtained by the study of analysis alone, we should use it. All the loci of analytic geometry fall under this heading. The definitions in terms of abstract lines, points and distances are quickly clarified by the mechanical constructions, given in most modern texts. Apparatus for making these mechanical constructions should be available in all college class rooms. Perhaps it may be added that to draw these curves by plotting isolated points, whose coordinates are obtained by substitution in the equation of the curve, is tedious, and not conducive to an understanding of the nature of the curve. This method of plotting points should not even be used in the high school.

All of us are acquainted with the fact, that while most students eventually learn how to find the derivative of a function, very few really understand what they are doing. How many of us really understood the Calculus when we studied it the first time? Now the limit of an infinite sequence of average rates over an ever decreasing interval of the independent variable, provided the limit exists, is an exceedingly abstract concept for beginners in Calculus. To help clarify the concept we make use of a falling body, or rolling object, measuring the distances and times, to find the velocity over smaller and smaller intervals. We also draw secants and a tangent to a graph, measuring the slopes. We set up tables of rates. All of these devices aid in giving meaning, and perhaps we do not use these and other devices in a sufficient degree to get real understanding.

When we come to the differential, a new element of trouble arises. Δy becomes dy, when we toss out terms in higher powers of Δx. Thus for the volume of a cube dV is $3x^2\Delta x$, but ΔV has in addition the terms $3x\Delta x^2$ and Δx^3. Then, in practical applications we say dV is a good approximation to ΔV if ΔV is sufficiently small. The manner in which this approximation becomes practical is comprehended more easily and quickly by the use of several models of $(a+b)^3$, where $a+b$ is always the same size, and b grows smaller, than by the substitution of numerical values. The final model may consist of three thin square cardboards of thickness b and length a and the cube $(a+b)^3$. The edges and corner $(3ab^2, b^3)$ are barely visible while the three square cardboards appear quite sizable. Thus we can discard $3x\Delta x^2$ and Δx^3, but not $3x^2\Delta x$ to get a very good approximation.

There are numerous places throughout the Calculus, e.g., the cycloids, evolutes, involutes, volumes of solids, etc., where the construction of the curves by the use of mechanical models saves words and time. In solid analytic geometry for all except the highly talented space perceptionists, models are indispensable. In higher mathematics, one need only consult the texts, and modern popular exposition

(such as "Mathematics and the Imagination") to note that drawings, models, and mechanical apparatus are used by talented mathematicians to explain the highly abstract analytic relationships.

Today, mathematics teachers in the high schools must know a great many practical applications of the mathematics they teach. Thus, while models and other aids are used throughout all the courses to clarify mathematical concepts, there is need also to show the students the practical application of the mathematics they have learned. It is the firm conviction of the writer, that a three or four semester hour course in the Application of Mathematics is a "must" for all high school teachers. Such a course would involve the applications of high school and college mathematics through the Calculus, to the explanation and use of modern precision instruments—modern computers and slide rule, the sextant, the transit, the solar attachment, the planimeter, the plane table, and an equatorial telescope. It would teach the elements of descriptive geometry, blue print reading, and elementary draughtsmanship. Application to electricity and radio; pulleys, belts and gears; tapers and cutting speed. It would require the students to construct simple instruments and devise problems in connection with their use of such instruments. The Seventeenth and Eighteenth Yearbooks are excellent source material for the course. Such a course would also teach that there is far more to mathematics than its own intrinsic beauty.

There remains the problem of additional training in the use of multisensory aids in the high-school instruction. These aids and the theory of value and place in the curriculum could be a part of the "methods" course. They can also be treated in part in courses on the history of mathematics, social uses of mathematics, trigonometry and solid geometry, assuming the last two courses are taught in college. Their use is also ingrained in the mind of the future teacher, if his own collegiate mathematical education has been enhanced by the continuous use of sensory-aids. This is one means of professionalization of subject matter in the training of teachers. Because of the breadth of the problem no general exposition of the high school phase of visual aids instruction can be considered here. I shall, in closing bring to your attention a proposal made almost a half-century ago.

The year after Perry made his address, Professor Eliakim Hastings Moore of Chicago University made his retiring address as President of the American Mathematical Society, using Perry's challenge as his theme. Under the title of "A Vision" Professor Moore said:

> *An Invitation*—The pure mathematicians are invited to determine how mathematics is regarded by the world at large, including their colleges of other science departments and the students of elementary mathematics, and to ask themselves whether by modification of method and attitude they may not win for it the very high position in general esteem and appreciative interest which it assuredly deserves:
>
> The fundamental problem is that of the unification of pure and applied mathematics.

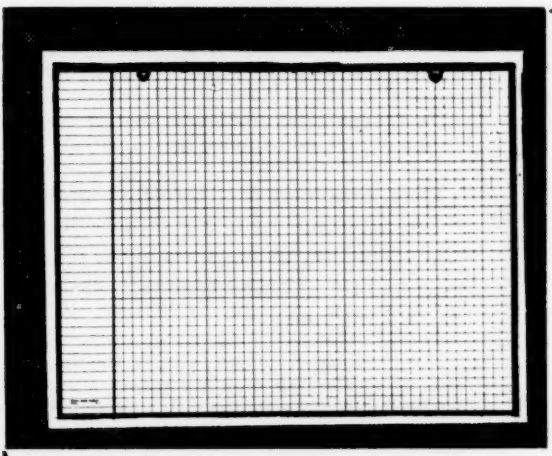
After a discussion of the silly outcomes of the attempts to teach supposedly abstract geometry and algebra in the schools, in separate compartments, Professor Moore continues:

> This program of reform calls for the development of a thoroughgoing laboratory system of instruction in mathematics . . . a principal purpose being as far as possible to develop on the part of every student the true spirit of research, and an appreciation, practical as well as theoretic, of the fundamental methods of science.

I, too, propose the question: Does not the ideal use of multisensory aids in the teaching of mathematics, lie in the mathematics laboratory? In the laboratory will be found all the measuring instruments, as well as material for constructing simple instruments; calculating devices of all sorts; geometrical models both fixed and with movable parts; models showing relations under variance, and also invariant properties; surveying and astronomical

instruments of all kinds; drawing boards and drawing instruments; charts and globes; and any additional devices that can be used to exemplify mathematical principles. The laboratory should be used to create a spirit of research and discovery. Field work should be a part of every mathematics course from grades three to twelve and perhaps in junior college. It should not be play, but must consist of planned experiments and testing of desired outcomes. Such laboratory work would never replace all the regular class room instruction in formal mathematics but would constitute a separate period as is the case in science teaching. It may be that ultimately, the physics and the mathematics laboratories could be combined into a thorough integration of these two subjects.

What I am saying is that laboratory teaching is one answer to giving reality to our subject without the loss of its abstract and theoretical aspects. There is not a single topic in grade or high school mathematics that cannot be exemplified and put to work in a mathematics laboratory. But so static and traditional have we been in our teaching, that it has taken the armed forces and industry to show us the values of laboratory techniques in aiding the teaching of mathematics. We must not open ourselves to attacks such as were made on the teaching of our subject before the war. We must not delay in making our teaching vital, real, and necessary.



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Teaching Thinking in Arithmetic

By H. G. Wheat
West Virginia University, Morgantown, W.Va.

There is nothing mysterious about thinking in arithmetic. All it takes is knowing what the question is and how to find the answer. Should you desire an exhibition of the activity, provide these two items of knowledge and let Nature take its course.

There is nothing mysterious either about teaching thinking in arithmetic. The teacher asks the question and the pupil tells the answer. He does, that is, if he knows it. If he doesn't know the answer, still all is well. He finds the answer. Then he tells it. It is as simple as that.

But the procedure is not self-starting. The teacher has to know what question to ask, when to ask it, and how to ask it. He must choose his questions wisely, keep them in order patiently, and ask them clearly. And, we may properly assume, the pupil does not know how to find answers, at least not expeditiously. But the teacher is supposed to know, and to have the patience and persistence needed to make the methods of finding answers clear.

The teacher-question—pupil-answer technique begins with intermediate steps which gradually disappear. At the outset, there are four steps: 1. The teacher asks; 2. The teacher shows how; 3. The pupil finds; 4. The pupil tells. Later, there are three steps: 1. The teacher asks; 2. The pupil finds; 3. The pupil tells. Finally, there may be but two steps: 1. The teacher asks; 2. The pupil tells.

The teacher-question—pupil-answer technique properly begins the first day of school arithmetic. To be fully successful, there is where it has to begin. The teacher does not say, "See, I have four apples." The resulting knowledge may be indisputable, yet its source is the teacher. Such knowledge is dependent knowledge. It is not as good, or as clear, or as useful, as the same knowledge when it has its source in the pupil. The latter kind is independent of the teacher. Instead of telling, therefore, the teacher asks, "How many apples do I have?" The pupil counts and tells. If he cannot count, the teacher teaches him how. Now he counts and tells. Through this procedure the pupil learns more thoroughly: what he knows he knows that he knows. He also learns more: he learns self-dependence.

The teacher-question—pupil-answer technique properly continues through the whole of school arithmetic. At no point does the teacher provide the answer. This is the pupil's business, his responsibility. He, the pupil, counts two and three together to get five. He counts six and eight together into a ten and four more. He separates eight to discover that there are four twos. He, much later, decides which of two numbers a situation is concerned about when the per cent is an issue. He makes mistakes. Let him. To err is human. Knowing what he looks to find, he has a gauge for what he finds. He profits from his errors.

The teacher-question—pupil-answer technique has certain distinguishing features:

1. *The teacher makes the question clear.* This the teacher does in either or both of two ways: a. The teacher makes clear the kind of *work* the question requires; b. The teacher makes clear the kind of *answer* the question requires.

To illustrate the first of the two ways, let us suppose the teacher's question is, "How many twos are in eight?" the first one of its kind. The teacher directs the pupil to count out eight objects, and next to count from the eight a group of two, then another, and another, and, finally, repeating the original question, to count the twos. Here, as elsewhere, the pupil tells the answer. He learns what the ques-

tion means in what he does to answer it.

To illustrate the second of the two ways to make the question clear, let us suppose the teacher's question is, "Four nines are how many?" the first one of its kind. There are several correct answers. "Nine fours" is one. "Three dozen" is another. "Six sixes" is another. But the teacher wants none of these. He wants the answer expressed in "*tens*." Since this is the answer he wants, he should so inform his pupil. To make his question clear, he must ask it in a way that reveals the kind of answer the question requires: Not "Four nines are how many?" but "Four nines are *how many tens?*" When the abbreviated form becomes clear, the abbreviated form can be used in safety.

2. *The teacher explains the method.* This the teacher does, as we have seen, to make the question clear. Conversely, he asks his question so as to make the method clear. To aid the clearness, he illustrates and demonstrates as he expounds. The teacher is the agency. He must be, since, at the outset at least, he is the one who knows the method and the question that requires the method.

How does the teacher know the method he should explain? Where does he look to find it?

The pupil's methods of work in learning arithmetic which the teacher should reveal are found in and suggested by the Hindu-Arabic number system. This is the single, unique system of number thinking that is in universal use in all civilized countries in the world. It is the system the pupil must learn and respect if he learns to think in arithmetic. The system records quantities to ten as single groups; larger quantities it records as *tens* and powers of *ten*. It makes *ten* the standard group for the evaluation of all quantities larger than nine. It uses nine numerals to record the smaller quantities. It uses the positions in which the numerals are written, with a zero to fill the empty places, to record the larger. The methods of the Hindu-Arabic system are the methods the teacher of arithmetic reveals to his pupil. As the system is the teacher instructs and the pupil does his work.

To illustrate, the teacher looks into the Hindu-Arabic system and notes the symbol zero as its unique feature, distinguishing it from all other numeral systems. He notes the single purpose and use of this characteristic symbol, namely, to hold position and to aid in keeping the numerals in their proper positions. He notes that the zero is not a numeral though it appears like one and seems to act like one. Accordingly, he frames his questions, as he comes to them, so as to guide the pupil to study the single function of the zero to keep positions clear. Conversely, the teacher avoids as he would the plague questions that suggest attention to usages of the zero that are not found in the Hindu-Arabic system, such usages, for example, as have for a generation confused both teachers and pupils, namely, the so-called "zero combinations," recent textbooks and courses of study to the contrary notwithstanding.

3. *The pupil assists the explanations.* To find different answers, the pupil pursues different methods. Yet a thread of similarity, of increasing visibility, runs through them all. At the outset, the pupil may not observe similarity; each new question requires a new method which the teacher must explain. Later, the pupil may begin to observe similarity; each new question requires a method he recognizes as much the same as those he has used. Now, he needs less of explanation from his teacher. What he understands suggests what he shall do next. Finally, when the pupil moves into the complexities of arithmetic, he may be relatively independent of his teacher's explanations.

To illustrate, dealing with tens the same as with ones requires considerable explanation when the addition of tens is first introduced. The subtraction of tens requires less explanation, provided the thread of similarity is shown and observed. The multiplication of tens, the division of tens,

and the multiplication by tens require progressively less and less explanation. Later, it may happen when division by two-place numbers is undertaken that instead of the teacher telling the pupil how to divide, the pupil will be able to tell himself. Finally, the pupil may transfer the general method of procedure to the handling of tenths and hundredths.

4. *The pupil does the work.* The teacher is merely the foreman, the supervisor, of the pupil's work, giving it direction and purpose. He keeps the pupil at work, on his toes, as it were, with increasing demands upon the pupil's attention. He carries the pupil's activity constantly, though gradually, to new levels of effort. He helps the pupil to avoid the frozen effects of mere repetition with diminishing attention. In the place of drill, he provides practice in thinking. He guides the pupil ahead, not "from the concrete to the abstract," but from simple abstractions to the more complicated, made so by the distractions of concrete situations.

Illustration can be drawn as readily from the pupil's early work as from his later. To make clear his division questions, for example, such as "How many twos are in eight?" the teacher at the outset may find it necessary to move a group of eight into twos or to render much aid in getting the pupils to make the re-arrangement of the objects. The pupil, of course, finds and tells all the answers, and thereby is quickly able to find answers to other division questions by carrying out the required re-arrangements himself. Later, the pupil can determine his answers by imagining the re-arrangements. Still later, he can provide his answers without resort to objects. All the while, due to his responsibility to come up with his own answers, the pupil is giving attention to division. He studies division, forms his idea of division, and develops his idea. Finally, as a further means of development, the pupil is given "division problems," that is, division questions that are involved in social and personal so-called situations. These are familiar situations and simple so as not to obscure too greatly the idea of division. Their purpose is to enlarge the pupil's idea of division through practice in recognizing division in somewhat familiar surroundings.

When the pupil has thus developed his idea of division, or addition, or average, or per cent, he is prepared to undertake a new type of study. He can now take up the study of situations, social and personal, that are somewhat new and unfamiliar, and in which the idea of division, or addition, or average, or per cent, is a characteristic feature. His ideas of number and of number relations open up the situations to his view and help to make them intelligible.

5. *The pupil provides the meaning and the teacher provides the expression.* The teacher all too frequently does things in reverse. He sees arithmetic much the same as the new style in automobiles: he has trouble telling which is front and which is back. Instead of providing for his pupil to develop number meaning, then carry it to the interpretation of social situations, he tries to get his pupil to draw number meaning from social situations, that can not be understood without it. He presents the symbols, then tries to tell what they stand for. He shows the process, then tries to explain it. He names the terms, then tries to supply the ideas that correspond. He impatiently gives the answer before his pupil knows what the question is.

This is not the way to do things and to get things done in arithmetic. As we have seen, the pupil derives his ideas from his work in seeking and supplying answers to fit questions that are clear. At these points he needs the terms and the signs to designate his ideas, to distinguish, to record, and to express them. The teacher obliges.

When the pupil is certain of the *four* apples he counts, the teacher introduces the figure 4, and has him learn it and write it as an expression of the idea. When the pupil has counted two and three together to get five, the teacher supplies the oral

and written language that states or records the activity: the sentence, "Two and three are five" and the written form of the figures 2, 3, and 5 in column. When the pupil has carried on many such additions, the teacher gives the activity its name.

Explanation of terms is not required. The teacher does not need to explain "addition" or "division" or "average" or "per cent." He merely furnishes the terms as names of activities and ideas already known to the pupil. Vocabulary difficulties get to be non-existent.

The procedure of naming the pupil's ideas when and as he gains them implies carefulness on the teacher's part to use the names correctly. The teacher does not call the numerals "numbers" and he does not thereby lead the pupil into the false study of numerals as substitutes for numbers. The teacher does not call the written figures 2, 3, and 5 in column a "combination" or an "addition" and he does not consider that addition is learned when the relation of the column to the sentence "Two and three are five" is committed. The teacher does not call sets of paired figures in column separated by horizontal lines "fractions" and he does not lead the pupil into a false study of these sets in place of fractions. The teacher, in short, does not consider the activity of number thinking the making and shoving around of numerals on paper or on the blackboard.

6. *The teacher asks his questions in ordered sequence.* The teacher desires his pupil to learn numbers, what they signify, both small and large. The teacher knows that learning is slow and step by step. He knows that successful learning is on-going. He questions accordingly, He seeks to guide his pupil's learning by guiding his pupil's study.

He guides his pupil's study of small numbers through counting one after another, through comparison of each with another, and through separations and combinations.

He guides his pupil's study of ten as the standard to which all numbers larger than ten are related.

He guides his pupil's study of larger and larger numbers through combinations and separations in relation to the standard ten and its powers.

He guides his pupil's study of the fraction as a relationship between numbers.

He guides his pupil's study of large numbers according to their fractional relations.

He guides his pupil's study of situations along the way in which the numbers, from smaller to larger, appear as characteristic features.

The teacher-question pupil-answer technique may have other distinguishing features. It has, we may be sure, the six that our discussion has suggested:

1. The teacher makes the question clear.
2. The teacher explains the method.
3. The pupil assists the explanations.
4. The pupil does the work.
5. The pupil provides the meaning and the teacher provides the expression.
6. The teacher asks his questions in ordered sequence.

The teacher-question pupil-answer technique stands up under the pragmatic test. It works. It works with beginning pupils and advanced pupils, with rapid pupils and slow pupils, with pupils who have succeeded and pupils who have failed. It also works with teachers!

To learn the methods that are required for number thinking and the questions that suggest the methods, the teacher has to study the number system. To learn the questions to ask first, how to make his questions clear, and how to explain the methods, the teacher has to study his pupil. To get to a question his pupil can grasp and to a method his pupil can understand, the teacher may have to break through many items of earlier inadequacies and failures. When he does thus reach far enough back, he is ready to begin and his pupil is ready to begin. With a question that is clear and a way to find the answer in mind, the pupil starts to move under his own power.

Arithmetic for the Twelfth Year

By Ralph C. Miller
Ridgewood High School, Ridgewood, N. J.

I will not go into detail concerning the teaching of a twelfth year arithmetic class, because you teachers know far more about that than I, but I will give you the development of the class in Ridgewood and its possibilities. What we are doing in Ridgewood is not new. Many schools throughout the country are giving the same work but under various titles. There are some who feel that our spade work in this branch of mathematics will be interesting to you. I believe the course fills a definite need in our school and community. The fact that the joint commission in 1940 suggested a social-economic arithmetic course for grade twelve or that the Commission for Postwar Plans in 1944 urged the high schools to include new and better courses for a large fraction of the school population, whose mathematical needs are not met in the traditional sequential courses, might be necessary to justify such a course in the eyes of some educators or college admittance officers, but not to me, nor to most of you. Even in a community such as Ridgewood, where 75% of the high school students plan to enter college, to contend that the sequential courses are the only ones of value and other courses may not be included in the curriculum is either due to pure stupidity or laziness. We either can't see beyond a few college requirements or knowing something else should be done for the students, we pass over the opportunity because of the work involved.

There are 1000 students in our Senior High School. The great majority take plane geometry and intermediate algebra. To a selected section of the best mathematics students we do as other schools in the United States are doing; we give the intermediate algebra and solid geometry in the eleventh year and the advanced algebra and trigonometry with a little analytical geometry and calculus in the twelfth year. This answers the question, "What do you do with the better student?" For the students not interested in College, General mathematics is given in the junior high during the ninth year. In the tenth year a course in mathematics as applied to the shop, and trades is given to students interested in this work, and in the eleventh year a course in "The Survey of High School Mathematics" is given.

Nowhere in our Senior High School previously was there given an opportunity for a student to get the mathematics he is likely to use most often in everyday affairs. As a senior, many of the girls know they will be married within the next two years. (In fact five of the girls in last year's class are now engaged!) Buying on the installment plan, borrowing money, saving money, and insurance, begin to have a meaning to both boys and girls. But you say these topics and many other important ones are given during the eighth year. That's right, they are, but surely you don't believe for a minute that because a girl gave the right answers then (and seemed terribly interested in the work) that she remembers anything about it when she's a senior! Just how interested is a twelve or thirteen year old youngster in taxes, insurance, installment buying, loans, bank discount, owning a home, stocks and bonds, and the like? Why just this year while on the subject of bank forms the seniors asked: "Do banks charge for check books?" "What is a counter check?" and "Does the bank charge you very much for the checks you write?" But a senior *is interested* because these are the words he reads about and hears spoken so very frequently. This seems to be the mathematics of everyday use to him. I would venture to say that such a course should be substituted for the intermediate

algebra of many of our students—and certainly should be taken, rather than solid and trigonometry by others.

The lady who has been secretary to the superintendent of schools for about fifteen years told me she hated high school mathematics, but she has had to work with figures and budgets everyday since. A lady visiting our school from a large picture studio in New York City told me the same thing. Can we say these ladies really got the mathematics they should have had if they hated high school mathematics? Of course, if you or I had been the teacher, they might not have hated it—as much.

Although we knew such a course would go far toward serving our needs, we were a bit shaky about the business of colleges accepting the course for high school credit. We wanted the course to be one to attract the good students, both college and non-college people; not one to be used as a dumping ground by the administration. We wrote to about twenty-five colleges giving in detail the contents of the course and asking if they would accept such for credit toward college entrance. You will be interested in the following replies:

COMMENTS ON ADVANCED ARITHMETIC BY VARIOUS COLLEGES

Tufts—Yes—for students interested in non-scientific and non-technical fields, the course should prove extremely valuable.

Bucknell—Your course is acceptable for entrance—but not as a substitute for algebra or geometry.

Brown—Will grant credit if H. S. accepts it for graduations: Prefer solid and trig for engineering or science.

Tusculum—Yes—Our committee is happy to recognize the fact that some attention is being given to courses of this kind.

Rutgers—Yes—the course will be a valuable contribution to gen. education and preparation for life experiences. I am pleased you are offering such a course.—Ch. of Math. Department.

Boston (Lib. Arts)—Yes

Lehigh—Yes

Montclair—Yes

Harvard—Yes—but each case should be considered separately. If a boy is to study science or engineering he should have math. through trig.

Middlebury—(Later letter)—We shall be glad to consider this course for entrance credit in individual cases where students do not have full schedule of college preparatory work.

Cornell—Yes—it could be taken as one unit from two electives outside our regular entrance list. We would advise College preparatory students to take solid geometry and trigonometry rather than this course.

Michigan—Yes, as one of the five electives—but not as a part of major or minor sequence in math. group.

N. J. C.—The com. has voted to accept this course for elective credit. It cannot be accepted instead of the usual algebra and geometry requirements, but may be offered as an elective entrance unit.

Smith—We offer no objection to granting entrance credit for course provided only that unit is accompanied by two full years of algebra and one of geometry.

Colgate—Yes—provided students have had geometry and intermediate algebra.

Skidmore—Yes—with algebra and geometry considered the usual offering in Math. to be included in the academic group of units.

Boston (Bus.)—Yes if the student has had algebra and geometry. I approve of H. S. introducing courses in higher arithmetic in Sr. year for students who are looking ahead in business in the same way that I approve of engineering students taking advanced algebra, trigonometry, solid in their senior year.

Does this mean the colleges are making an "about face" and now will give entrance credit for courses in general mathematics? Certainly not. But it does mean that for those pupils who are college material (those capable of taking algebra and geometry) an advanced arithmetic course is considered as valuable as most of the other courses and for some college people it seems of greater value.

The college board people put no barrier in the way for this arithmetic course in the twelfth year. Except for the boys going into engineering schools, the students take the aptitude test on English and mathematics. And this test is usually taken at the end of the eleventh year.

Two years ago this spring we added the subject in the twelfth year. After a survey had been made of the election for the following year, only four students had signed for the course. I then stepped into the picture and did some missionary work. When the schedules were made during the

next month, twenty-seven students had elected the arithmetic course (ten from my own home room). When school began in the fall, there were thirty-six in the class and no standing room permitted. One-half of the class had had a year of geometry and two years of algebra. But as it turned out, only in rare instances could you have been sure which students had had the greater mathematics training because there was an interest in the work. In fact, a girl who had failed elementary algebra in her first try at it, made the highest mark on the unit of statistics. She had talked to me previously to signing for the class. It seemed this was just what she needed but she was afraid to try it because of her previous record. There was no harder worker than she during the year. At the end of the course she came to me and thanked me for insisting that she tackle the work.

Probably no book will satisfy the needs of the course. Every applied mathematics text that contained work in geometry and alegebra, I did not consider. I wanted to get away from all of that, except to fit it into my course at some needed point. The text we now use fairly well meets our needs. The material in the review of arithmetic fundamentals, problem solving, per cent, installment buying and loans, insurance, taxes, various consumer problems, simple and compound interest, saving money, and work on areas and volumes is all excellent. A good beginning is made on annuities, but here you will largely be on your own to make up your ordinary annuity tables and problems. A page or two is given to linear measure using the metric system. But you won't want to stop with that in this important work. To know that the metric system is a decimal system is valuable but certainly no more so that the linking together of mass. volume, and capacities. That 1200 c.c. of water will weigh 1200 grams or 1.2 k.g. and be equal to 1.2 liters is the interesting and important principle here in educating these boys and girls. Of course a good knowledge of physics or chemistry will help to put this unit over, but even then don't be discouraged if on a test some pupil says that he weighs 400 grams and is 60 centimeters tall! The book I use does nothing with the valuable unit of statistics, aside from taking a few pages to find average and medium; so the planning of this unit was left to me. I found the students liked this work. I prefaced the unit by one on graphs. Being treasurer of the Athletic Association, I compiled quite a bit of data about the finances of the organization over a period of 12 years. The class assembled the data on several kinds of graphs. We then went into work on central tendency, deviation from the mean, the normal curve and some of its properties. Standard deviation and the constant 0.6745 became a part of their vocabulary for a while anyway.

I would like to outline the work of the year and then touch briefly on a couple of items.

The work consisted of:

1. Review of arithmetic fundamentals.
2. Measurement of areas and volumes as used in everyday life.
3. The metric system.
4. Problem solving, emphasis here on understanding problems of interest to the consumer.
5. Problems in per cent.
6. Simple interest, discount and bank discount.
7. Installment buying and borrowing of money.
8. Problems of the consumer—as, budgets, taxes, owning a car or a home.
9. Saving and investing money.
   a. Compound interest
   b. Annuities
   c. Building and Loan Companies.
10. Insurance
11. Taxation
12. Units like logarithms and permutations, found in any algebra text.
13. Statistics

Now this outline is wide open for improvement—take out some material—add other units on scale drawing, the slide rule, stocks and bonds, computing with approximate data, etc.

You may want to touch lightly on *the mathematics of life* insurance, but the

pupils are at the age they are about right for the salesman. But again don't get discouraged when at the end of the unit some of the students believe the premium on term insurance is the highest and that the endowment policy is the one needed most by a young man with a family!

May I say a word about the teaching of one part of the course without boring you? Installment buying and borrowing money from personal loan companies should be shown to be the same thing and therefore the same formulas applied. You either make payments to the store or to the loan company. A junior high school principal in our village was surprised that there was a formula by which you could find the rate of interest charged by Sears Roebuck & Company when you use their "Time Payment Plan." So our formula

$$r=\frac{24C}{P(n+1)}$$

for monthly payments, *is important.* He also thought $\frac{1}{2}$ of 1% interest per month on the unpaid balance was different than 6% per year!

For annuities and compound interest work, four tables are given to students; the amount and present valuable tables for both compound interest and ordinary annuities. I guarantee you will get much pleasure from teaching this unit if you honestly attempt to give the tables meaning and point out the dove-tailing of these tables. I thought I had clinched the understanding one day—but the class asked on the three successive days that I once more explain the tables. A girl's face brightened on the last day and she said, "STOP." "I have it, now don't get me mixed up again!"

But the work pays dividends. A senior told me he just taught an economics class how a note is discounted at a bank—*and explained it so well even the social studies teacher understood!* Or when a girl comes to you and says the problems in annuities are even too difficult for her dad but she understands them or the slower ones who admit they never before understood how to get the answers to such simple problems as, "If a man makes a third of the cost by selling an article for $6.00, what was the cost?" Or even more satisfying the experience of having a teacher in the department tell you she had a girl in her plane geometry class and she couldn't add two and two, but now in my senior work this girl is working in study period as never before!

If such a course in arithmetic as I have outlined is to find its worthy place in the high school, then it will probably be through the effort of all the younger teachers. You teachers who have had college courses in mathematics of statistics and finance as a part of your training surely have just what it takes to teach and motivate this new arithmetic. The older teachers will gladly hand over to you the job of planning and teaching the course. In fact they will bribe you to take the class. You can then take the driver's seat and put the class in its rightful place in the school.

In due time, I believe your class will become one of the more important mathematics sections in the school because you will be teaching the mathematics the students will likely use.

Increase in Price of Yearbooks

The National Council is compelled to increase the price of the 15th and 16th yearbooks to $2 each, postpaid, effective May 1, 1947. The 17th and 18th yearbooks are also $2 each postpaid, but the new 19th yearbook is $3 postpaid. We are sorry to have to increase our prices, but we hope that our members will all secure increases in salary for the coming year, so that our increases will not be burdensome. In the days ahead it is extremely important for every teacher of mathematics in the elementary and secondary schools to show his allegiance to The National Council by keeping up his membership in that organization. We know you will not fail us.—EDITOR

The Logic of the Indirect Proof in Geometry

Analysis, Criticism and Recommendations[1]

By NATHAN LAZAR
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THERE are few topics in plane geometry that have had as strange a career as that of the indirect proof. In Euclid's Elements it does not appear separated from the other method of proof. There is no comment indicating that it is a unique device. There are no reasons quoted for each of the steps taken, as is done so meticulously in the other proof. Moreover, this type of mathematical reasoning invented by Eudoxus in 375 B.C. had become so much a part of the methodology of the Greek geometers that Euclid uses it no less than 14 times in 35 proofs in Book I, and 16 times in 31 theorems in Book III of his Elements.

Philosophers, logicians, mathematicians, commentators, textbook writers, teachers, and, of course, pupils, have expressed dissatisfaction with this method of proof. The reasons given ranged from the fundamental one, casting doubt on the logical validity of this proof to the naive complaint of the immature student that the proof is not convincing because it "beats around the bush."

As long as Euclid's Elements was a subject of study for mature men or, at best, for students pursuing a bachelor's degree in a university or at a college, little was done to explain the rationale of indirect proof. When, however, geometry became a part of the curriculum of the secondary school, and both textbook writers and classroom teachers became more sensitive to the needs of the young students, explanations and simplifications of this method of proof began to appear in the pedagogic literature as well as in the books given to students.

There is another factor that has operated in recent years to focus the attention of educators on this method of proof. The doubts cast on the value of geometry for general education has forced teachers and writers to look around for those topics in geometry that have transfer value to non-geometric situations. They found in this topic a highly fruitful source of examples to afford opportunities in training for reasoning. As a result of this tendency, a highly paradoxical situation has arisen. The actual number of geometrical theorems that have to be demonstrated by the indirect method has dwindled greatly, almost to the vanishing point; while the number of non-mathematical statements to which this method is applied has increased by leaps and bounds.

Despite the prominent place that this method has been occupying of late, and despite many attempts at clarification, it still remains a topic that is approached by many teachers with a great deal of timidity and is taught with very little pleasure.

Without attempting an exhaustive analysis of all the factors that may be responsible for the above attitude of mind, I should like to point out a few personal dissatisfactions with this method and to offer another approach which *may* make it more acceptable both logically and pedagogically, to teachers and pupils alike.

LACK OF DEFINITION OF "DIRECT PROOF" AND "INDIRECT PROOF"

That there is a great deal of confusion about the nature of indirect proof can be

[1] Read, in part, at the general meeting of the National Council of Teachers of Mathematics, at Atlantic City, New Jersey, on February 28, 1947. This paper is a summary of a more detailed study on the same topic which is in preparation. Reprints *with covers* may be had for 25¢ each postpaid from THE MATHEMATICS TEACHER, 525 W. 120th St., New York 27, N. Y.

evidenced by the fact that very few books take the trouble to give a definition of the crucial terms, "direct proof" and "indirect proof." Moreover, there is no agreement among authors whether "proof by forced coincidence"[2] is to be considered "direct" or "indirect."

But not only is there no unanimity among various authors as to whether a certain kind of proof is to be called direct or indirect, but often the same author is not consistent with himself. In the earlier pages of his book one writer defines indirect proof so as to exclude "proof by forced coincidence" from that category, whereas later in the book he refers to that type of proof as *indirect proof.*

Contrary Statements *vs.* Contradictory Statements

Another source of confusion is the wrong conception of the nature of contradictory statements, especially in reference to *implicative propositions.* As is well known, two statements are contradictory when the following two conditions are satisfied:

I. If either is true, the other must be false, and

II. If either is false, the other must be true.

Thus if a and b are numbers, the following statements are contradictories of each other:

$$a = b$$
$$a \neq b.$$

If a and b are lines, then the following two statements are contradictories:

a is co-planar with b
a is skew to b.

Similarly if a and b are lines and coplanar, then the following two statements are contradictories:

a is parallel to b
a is not parallel to b.

In general, to form the contradictory of a given relational statement, deny the existence of the relation.[3] But the following two statements are not contradictories of each other:

$$a > b$$
$$a < b.$$

Although condition I is satisfied, condition II is not—since if one of these two statements is known to be false, there is no guarantee that the other must be true. It is possible that both may be false, as would be in case where $a = b$.

In logic, both traditional and modern, the special term "contraries" is applied to the statements having these properties. Thus, the statements

$$a = b$$
$$a > b$$
$$a < b$$

are contraries of each other. If one of these is true, the other two must be false. But if one of the three is false, we do not know which one of the other two is true. It should be observed that whereas by definition contradictories come in pairs, there is no limit to the number of statements that may be contraries of each other. Thus, if a and b are human beings, the following statements are contraries of each other:

a is the father of b
a is the uncle of b
a is a cousin of b
a is a brother of b
a is a son of b.

It is of course possible that all of the above statements may be false if it is the case that a is not related to b.

[2] The description "proof by forced coincidence" is used to refer to the usual proof of a theorem such as "If a line divides two sides of a triangle into proportional segments it is parallel to the third side." One form of the proof consists of constructing through one of the two points of intersection, a line that is parallel to the third side, and then showing that the line that was there all the time must perforce coincide with this line newly drawn.

[3] The writer is purposely avoiding as much as possible the use of the technical term "proposition" as used in contemporary logic because of the possible confusion with its meaning in geometry.

Implicative Statements and Their Contradictories

Nearly all, if not all of the theorems of mathematics in general, and of geometry in particular, are, or can be put, in the form of an implicative statement, of the type

$$h_1h_2h_3 \cdots h_n. \text{ imply } c_1c_2 \cdots c_m$$

where h_1, h_2, h_3, $\cdots h_n$ represent respectively the statements of the hypothesis and $c_1,c_2 \cdots c_m$ the statements of the conclusion.[4]

Unfortunately it is not a very simple matter to form the contradictory of an implicative statement. Even so subtle a thinker as Wentworth[5] went wrong in his illustration of contradictory statements, though he was right in his definition of them.

> The contradictory of a theorem is a theorem which must be true if the given theorem is false and must be false if the given theorem is true. Thus
>
> 1. A theorem: If A is B, then C is D.
> 2. Its contradictory: If A is B, then C is not D.

The same error was also committed by Schultze in his highly influential and very popular handbook for teachers,[6] with the result that it has been widely copied and spread in many magazine articles, manuals and textbooks.

In order to show that the above statements given by Wentworth are not contradictories, all that is necessary is to give some illustrations which although they adhere to the pattern given above, do not, however, fulfill the conditions of the definition.

Consider the following statements:

1′. *If two parallel lines are cut by a transversal, the corresponding angles are right angles.*

2′. *If two parallel lines are cut by a transversal the corresponding angles are not right angles.*

Evidently 1′ and 2′ follow the patterns of 1 and 2 given by Wentworth. Moreover, 1′ and 2′ are both false, since from the statement that "two parallel lines are cut by a transversal" it is impossible to deduce either that the corresponding angles are right angles, or that they are not right angles. Since 1′ and 2′ are both false, they cannot be contradictories of each other, and it must be concluded that the patterns set by Wentworth are incorrect.

Other sets of statements, both of which are false, can be given that fit the pattern but not the definition:

3a. *If a figure is a quadrilateral, the opposite angles are supplementary.*

b. *If a figure is a quadrilateral, the opposite angles are not supplementary.*

Again the above statements are contraries of each other, but cannot be contradictories, since, as a matter of fact, they are both false.

The problem still remains: How to contradict an implicative statement of the type so prevalent in geometry—"if . . . and if . . . and if . . . , then. . . . "

But before that can be shown, two more concepts will have to be introduced which are in constant use in daily life, in logic, and in philosophy of mathematics, i.e. consistent statements and inconsistent statements.

Consistent Statements

Consistent statements are such that can be simultaneously true, such as

4a. ABC is a triangle

b. $\angle A = 70°$

c. $\angle B = 50°$

or

5a. $ABCD$ is a parallelogram

b. E is a point within the parallelogram

c. EA bisects $\angle A$.

d. EB bisects $\angle B$.

Evidently the conditions in the hy-

[4] For convenience, the above general form will be abbreviated hereafter into $h_1h_2h_3$ imply c. Moreover, the symbols $\bar{h}_1$, $\bar{h}_2$, $\bar{h}_3$, and $\bar{c}$ will be used to refer to the corresponding contradictories of h_1, h_2, h_3, and c.

[5] Wentworth, C. A., *Plane Geometry*, Boston, Ginn and Co., 1902, p. 5.

[6] Schultze, Arthur, "*The Teaching of Mathematics in Secondary Schools*," New York, The Macmillan Co., 1912, pp. 177–178.

pothesis of *every* theorem in geometry—and in all other mathematical disciplines—must be consistent, before true deductions can be made from them.

Inconsistent Statements

Inconsistent statements are such that cannot possibly be true at the same time. Each of the following is an illustration of an inconsistent set of statements:

6a. I travelled 150 miles by automobile.
b. I left my hometown at 9 A.M.
c. I arrived at my destination (in the same time zone, on the same day), at noon.
d. I never exceeded the legal speed limit of 40 miles per hour.

7a. ABC is a $\triangle$
b. $x = x'$
c. $\angle B$ is opposite x
d. $\angle C$ is opposite x'
e. $\angle B \neq \angle C$

8a. $ABCD$ is a quadrilateral
b. $BC \| AD$
c. $BA \| CD$
d. $BC \neq AD$.

What Does an Implicative Proposition Assert[7]

Before an implicative proposition can be contradicted, its full meaning must be ascertained. What are we asserting when we say in demonstrative geometry for example, that "if two sides of a triangle are equal, the angles opposite these sides are also equal?" We are certainly saying *more* than the bare fact that whenever a triangle has two equal sides, it also has two equal angles, and that these angles are opposite the equal sides. What we are asserting is that the latter statements *are implied* by the former. To put it in another way, we are asserting that the equality of the angles *is deducible from* the equality of the sides, and from certain other theorems, definitions, and postulates that have preceded this proposition.

A little reflection will also show that the original statement also means that you *can not* assert simultaneously the two sentences that "a triangle has two equal sides" and that "the opposite angles are *not* equal." In other words the statement "a triangle has two equal sides" *is inconsistent with* the statement "the opposite angles are not equal."[8] Or more generally the statement

$h_1 h_2 h_3$ implies c

is equivalent to the statement

$h_1 h_2 h_3$ and $\bar{c}$ form an inconsistent set.

How to Contradict an Implicative Proposition

There is more than one way of contradicting an implicative proposition, but none of these contradictories can possibly be another implicative statement. The above analysis of the equivalence between an implicative statement and an inconsistent set affords us, however, a simple method—on a secondary school level—for contradicting an implicative statement: deny the inconsistency between $h_1 h_2 h_3$ and $\bar{c}$ by affirming that $h_1 h_2 h_3$ *are consistent* with $\bar{c}$.

Thus the contradictory of the statement

[7] It should be noted that throughout this paper, the following restrictions are made on the conditions under which a true statement is called implicative:

1. Each of the statements in the hypothesis must be independent of each other.

2. None of the conditions in the hypothesis can be superfluous; the failure to observe this restriction is commonly called "overloading the hypothesis."

Thus the following proposition violates the first restriction:

If a line joins the midpoints of two sides of a triangle and is parallel to the third side, it is equal to one half of it.

On the other hand, the second restriction is violated by the following proposition:

The lines joining the midpoints of the sides of a parallelogram, taken in order, form another parallelogram.

The same two restrictions will also apply to a set of inconsistent statements.

[8] Every implicative statement has, of course, its contrapositive equivalents. But this is not the place to expatiate on that subject, since it has already been discussed by the writer in his articles in this Journal. "The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry," The Mathematics Teacher, Vol. 31: 99–113, 156–174, and 216–240, March, April, May 1938.

If two sides of a triangle are equal, the angles opposite are equal

would be

the statement that "*two sides of a triangle are equal*" *is consistent with the statement "the opposite angles are not equal.*"

Similarly the contradictory of

if two lines are cut by a transversal, and if a pair of corresponding angles are equal, the lines are parallel

would be

the two statements "*two lines are cut by a transversal,*" *and "a pair of corresponding angles are equal" are consistent with the statement "the lines are not parallel.*"

Evidently the last two contradictories are false, since the originals are true.

As a simple test of the validity of this procedure of contradicting an implicative proposition let us apply it to the contrary statements used on page 227.

Let us imagine that someone asserts the (false) statement that "if two parallel lines are cut by a transversal, corresponding angles are right angles." His opponent wishing to contradict him would—according to the analysis given here—then say that the statement "two parallel lines cut by a transversal" *is consistent with* the statement "the corresponding angles are not right angles." The latter statement is evidently true, thus fulfilling the conditions of the definition of contradictories.

Similarly the contradictory of the (false) statement

If a figure is a quadrilateral, the opposite angles are supplementary

would be

the statement "*a figure is a quadrilateral*" *is consistent with the statement "the opposite angles are not supplementary*"

The latter statement is evidently true.

It should be observed that, from the point of view of the application to mathematical reasoning, an implicative proposition may be false for either one of *two* reasons:

(a) The hypothesis and the conclusion are independent of each other, i.e. there is no necessary connection between the statements in the hypothesis and those in the conclusion. Thus in the (false) statement stated above, there is no necessary connection between a figure being a quadrilateral and its opposite angles being supplementary. In such cases the hypothesis is consistent *both* with the (so called) conclusion—"The opposite angles are supplementary"—and with the contradictory of the conclusion—"the opposite angles are not supplementary."

(b) An implicative statement may also be false because the (so-called) conclusion is the contradictory of the conclusion which can be deduced from the set of hypotheses. In this case the hypothesis—"the equality of a pair of corresponding angles"—is not consistent with the conclusion—"The lines not being parallel," but it is consistent with the denial of the conclusion—"the lines are parallel."

Thus the logical pattern for contradicting an implicative statement that is recommended here—affirm the consistency of the hypothesis with the denial of the conclusion—is suitable for true implicative statements as well as for the two types of false implicative statements.

How to Show That a Set of Statements is Consistent

Although there are few occasions in secondary school algebra for discussing or exhibiting the consistency of a set of statements, it might be pedagogically sound to make students conscious of this logical relation if for no other reason than to arouse their curiosity and prepare the ground for a discussion of the *inconsistency* of a set of statements. Strangely enough it is only in the *second* course in algebra that these concepts occur explicitly in connection with "consistent and inconsistent equations." These come up in the more general topic of the graphing of sets of linear equations, and in the determining of conditions under which such

sets of simultaneous equations have a common solution.

The occasions for pointing out sets of consistent statements in deductive geometry are, however, very numerous. As pointed out above, the conditions in the hypothesis of *every* true geometrical proposition must at *least* be consistent with each other if they are to imply the claimed conclusion. Moreover, in every true implicative statement the conclusions and the conditions in the hypothesis also form a consistent set.

It might be desirable then to give students practice in analyzing sets of statements such as the following:

a AB and CD are straight lines
b They intersect in E
c $\angle AED = 70°$
d $\angle CEB = 70°$.

It should also be pointed out that although c and d are consistent with a and b, they are not deducible from them; but that either c or d is deducible from the remaining three statements.

The concept of consistency is so simple that it is almost unnecessary to prescribe tests for its determination. If, however, the students insist upon an explicit formulation, the following two procedures may be employed

1. Experimental method—with the help of accurate instruments (including especially the non-Platonic protractor and graduated straight edge) construct a careful diagram, fulfilling all the conditions given. If it is possible to do so, the conditions are consistent.

2. Another possible method is to have the students "examine" the individual statements and see whether, in view of the laws of geometry they already know, it is possible for these statements to be true at the same time.

This method is, of course, unreliable.

Assume that a class in plane geometry is following the traditional sequence wherein the theorem about the equality of the base angles of an isosceles triangle precedes the theorem about the sum of the interior angle of a triangle and its corollaries—especially the one that it is impossible for a triangle to have more than one obtuse angle. In the early weeks of the course most of the students in that class will—in the absence of a diagram—most probably judge the following set of statements to be consistent:

a ABC is a triangle
b $AB = AC$
c $\angle B$ is an obtuse angle.

However, the possible intellectual harm that may result from the use of either method should not be exaggerated. For, in the first place, it merely shows that when a student is in possession of relatively few facts of a deductive system he should not rely on his "impressions." In the second place, it also illustrates the point that perhaps a simple experiment, even with crude instruments, has its rightful place in a course of deductive geometry. Thirdly, any attempts at making deductions from the above set will pretty soon come to a futile end, with the consequent suspicion that something is logically wrong with that set of statements.

How to Show That a Set of Statements is Inconsistent

There is a similarity between the methods indicated above in showing that a set of statements is consistent, and the methods used to prove that a set is inconsistent. For here, too, two methods are available, the experimental and the logical.

1. In the experimental method an attempt is made, by means of careful drawing and measurement, to fulfill all the conditions set forth in the statements. An impasse is quickly reached, which indicates the possible incompatibility of the set of conditions. This method, though it is not the kind to be used ultimately, is, however, desirable from a heuristic and a pedagogical point of view, since it forces the student to come to grips with the psychological meaning of inconsistent statements. It also resembles the usual deductive proofs where many experienced

teachers begin with an experimental approach, in order to clarify the meaning of the theorem and proceed therefrom to the proper task of showing that its truth is deducible from the given hypothesis and the preceding propositions and postulates.

2. The second method—the logical—consists in operating with *every one* of the statements in the set, and deducing from them another statement which is false, because it is the contradictory or the contrary of a postulate or of a proposition which has already been shown to be true.

A variant of this method consists in deducing from *all but one* of the statements of the set the contradictory or a contrary of the remaining one.

Let us apply this technique to the illustrations given above, p. 228, ex. 6. From 6b and 6c we can deduce that he travelled

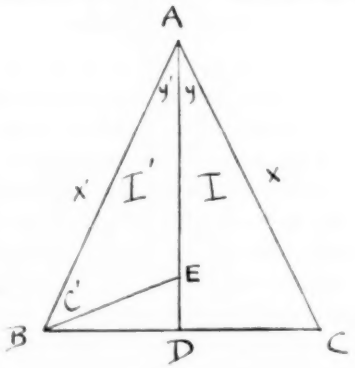

FIG. 1

3 hours. If this is combined with 6(d) we can infer that he travelled no more than 120 miles. If this fact is combined with the 6(a) we get the false "equation 120 miles (or less) = 150 miles.

Since the joint assertion of 6a, b, c, d leads to a false statement, the set is therefore inconsistent.

Example 7 on page 228 will perhaps be of more immediate interest since it deals with a geometrical situation:

Given: a. ABC is a $\triangle$
b. $x = x'$
c. $\angle B$ is opposite x
d. $\angle C$ is opposite x'
e. $\angle B \neq \angle C$.

To Show: That statements a, b, c, d, e, form an inconsistent set.

Plan: Use every one of the above statements in a process of reasoning and deduce from them a false statement.

Proof: 1. Let AD bisect $\angle A$
2. Since $\angle B \neq \angle C$ one of them must be larger
3. Assume $\angle B$ is the larger one
4. On it lay of $\angle C'(ABE) = \angle C$
5. In $\triangle I'$ and I
a $\angle C' = \angle C$
b $x' = x$
c $\angle y' = \angle y$.

Therefore $I' \cong I$

$\therefore AE = AD$.

The last statement is obviously false, since it violates the postulate that the whole of a line is larger than any portion of it. We have thus shown that a false statement is deducible from statements a, b, c, d and e, and have thereby proved that that set of statements is inconsistent.

There is of course an easier method of showing that the above set is inconsistent. The first four statements, a, b, c, d, yield, —by the usual method that is used in proving that the base angles of an isosceles triangle are equal,—that $\angle B = \angle C$. This statement combined with e, that $\angle B \neq \angle C$, yields $\angle C \neq \angle C$, which is of course a violation of the Law of Identity. The other method was used above, because it resembles more closely the method of indirect proof to which these analyses are an introduction.

Let us illustrate the former technique of exhibiting the inconsistency of a set of statements by using one more example shown above, p. 228, ex. 8.

Given: a. $ABCD$ is a quadrilateral
b. $BC \| AD$
c. $BA \| CD$
d. $BC \neq AD$.

To Show that the above set of 4 statements is inconsistent.

Plan: Show that a false statement is deducible from the above set of statements.

Proof: 1. Since $BC \neq AD$, let AD be the longer side
2. On AD lay off $AD' = BC$
3. Draw CD'
4. Draw AC
5. In $\triangle ABC$ and $AD'C$
   a $BC = AD'$
   b $BC \| AD' \therefore \angle x = \angle x'$
   c $AC = AC$
   d $\therefore ABC \cong AD'C$
6. $\therefore y = y'$
7. $\therefore CD' \| BA$
8. but $CD \| BA$.

We therefore have two lines CD' and CD through one point C, both parallel to a third line AB. This contradicts the well known postulate, that through one point outside a given line only one line can be drawn parallel to the given line. We have

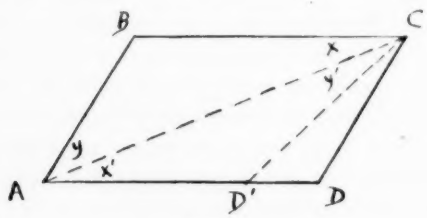

FIG. 2

thus shown that the original set of four statements is an inconsistent one since it led to a false statement.

Here too, a simpler method was available for showing the inconsistency of the set of statements. But the purpose here too, as in the preceding illustration, was to adumbrate the similarity of patterns between inconsistent sets and indirect proof.

THE RELATION BETWEEN AN INCONSISTENT SET OF STATEMENTS AND AN IMPLICATIVE PROPOSITION

The reader has undoubtedly noticed that each of the three illustrations used above can be transformed from an inconsistent set to an implicative set wherein the contradictory of the last statement is deducible from all the preceding ones. Thus Example 7 becomes the familiar theorem "if two sides of a triangle are equal, the angles opposite these sides are also equal." Example 8 is, of course, the theorem, "in a parallelogram the opposite sides are equal."

The possible equivalence between an inconsistent set of statements and an implicative statement[9] will now be postulated as follows:

I. If $s_1, s_2, s_3 \cdots s_n$ are inconsistent then the contradictory of any one of the statements is implied by the joint assertion of the others.

II. If $h_1h_2h_3$ imply c, then the joint assertion of $h_1h_2h_3$ and $\bar{c}$ form an inconsistent set. In other words the statement

$h_1h_2h_3$ imply c

is equivalent to

$h_1h_2h_3\bar{c}$ are inconsistent.

APPLICATION OF THIS EQUIVALENCE TO INDIRECT PROOF

A careful study of the indirect proof as used by Euclid in his Elements and by other mathematicians in their respective fields of endeavor has convinced this writer that the underlying rationale behind this method of proof is, in the overwhelming majority of cases, the mutual equivalence between an implicative proposition and an inconsistent set.

Perhaps the best way of clarifying this is to exhibit a familiar indirect proof in this new logical garb.

If a line divides two sides of a triangle proportionally it is parallel to the third side.

Given: a. $\triangle ABC$
b. B' divides AB into x and y
c. C' divides AC into x' and y'
d. $\dfrac{x}{y} = \dfrac{x'}{y'}$.

To Deduce:
e. $z \| z'$

Plan: We do not know—or we do not want to use—the method of deducing the

[9] See note 4 on p. 227 and note 7 on p. 228.

parallelism of $B'C'$ and BC from the given data, and from the theorems and postulates that have preceded this theorem. We shall therefore establish the truth of the above theorem by showing the truth of another one which is equivalent to it, i.e., the inconsistency of the following statements:

a ABC is a $\triangle$

b B' divides AB into x and y

c C' divides AC into x' and y'

d $\dfrac{x}{y}=\dfrac{x'}{y'}$

$\bar{e}$ z is not parallel to z'.

Proof (of the equivalent theorem)

1. Since according to $\bar{e}$ the line z' is not parallel to z, let us construct a line z''

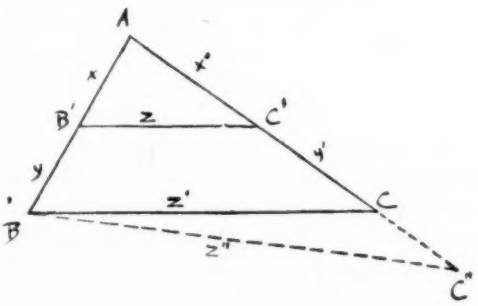

FIG. 3

through B parallel to z, intersecting the prolongation of AC in C'' (The proof will remain unchanged if C'' falls between C' and C)

In the new triangle formed, ABC'', $z \| z''$

$$\therefore \quad \frac{x}{y}=\frac{x'}{C'C''}.$$

Since

$$\frac{x}{y}=\frac{x'}{y'}$$

$$\therefore \quad y'=C'C''.$$

Since the last statements violate the postulate that the whole of a line segment is greater than any portion of it, we have thus shown that the set of statements a, b, c, d, $\bar{e}$ is inconsistent, and have thereby shown *indirectly* that the statements abcd imply e. It should be noticed that the construction lines and most of the steps in the above proof are almost the same as can be found in most textbooks. The *logic*, however, is different.

THE ADVANTAGES OF THE ABOVE APPROACH

1. Perhaps the most important argument in favor of this approach to indirect proof is the admission on part of the teacher or textbook that no attempt will be made to deduce the "conclusion" in the usual way from the statements in the hypothesis, but that a substitute proposition will be proved. The student—the mature as well as the immature—is not therefore bewildered by what seems to him a round-about way of getting a proof. He knows the exact logical status of all the elements in the proof.

2. Of late, the tendency has arisen to use the indirect proof only in cases where direct proof is impossible or very difficult. The students have therefore formed an impression that the method of indirect proof is less general or even less reliable and certainly less convincing than direct proof.

The method advocated here, because of the simplicity of its rationale, shows that the indirect method is the more general method. *Every* theorem in mathematics can be proven by the indirect method, since every theorem can be transformed into an inconsistent set by the simple expedient of asserting simultaneously the original set of hypotheses together with the contradictory of the conclusion.

3. Because of the clarity of its objective and the simplicity of its technique, this method has one further advantage of encouraging the immature student to try the indirect method and to realize its possibilities. As is well known, the indirect method of proof is used a great deal in advanced mathematics. It has been noticed by many teachers of the advanced courses that students do not feel at home

in that method as they do in the direct proof. This unfamiliarity and hesitancy is undoubtedly due to a lack of practice as well as to a lack of understanding of the logical basis on which it rests.

4. The indirect method of proof is also known by its Latin name as *Reductio ad Absurdum*. The reason for this name becomes clear in view of the analysis shown here, since the original theorem is "reduced" (i.e., changed) to an absurdity (i.e., an inconsistency).

5. Some of the arguments given against usual indirect proof are that it often leads to a distorted and inexact figure, that it assumes that to be true which is obviously untrue, and that it requires that to be constructed which is actually impossible. From the point of view of the analysis recommended here these are not arguments *against* the indirect proof but *constitute the very nature* of indirect proof. The objective of the method is to show that the statements are incompatible and irreconcilable. The student is therefore neither surprised at the impossibility of constructing an accurate diagram nor at the assumption of the falsity of the conclusion.

Other Methods of Indirect Proof

An attempt will be made here to present very briefly two other methods of indirect proof. The purpose is twofold:

(a) not to leave the impression that the method of inconsistency is the only possible approach to indirect proof;

(b) to prepare a more adequate background against which to present a critique of the analysis of the nature of indirect proof that is current in many textbooks, teachers' manuals, and magazine articles.

The *second* method depends on The Law of Contraposition which states in part that the theorem

$h_1h_2h_3h_4$ implies c

is equivalent to

$h_1h_2h_3\ \bar{c}$ implies $\bar{h}_4$.[10]

Perhaps the best way of illustrating this technique is by applying it to the same theorem which was used above for the method of inconsistency.[11]

The diagram, the "given" and "to deduce" are, of course, the same. The "plan" and "proof" are, however, different.

Plan. To show the truth of the theorem, abcd imply e by showing the truth of its contrapositive equivalent,

a b c $\bar{e}$ imply $\bar{d}$.

The equivalent contrapositive will, therefore, be as follows:

Given: a. ABC is a triangle
b. B' divides AB into x and y
c. C' divides AC into x' and y'
$\bar{e}$. z is not parallel to z'.

To Deduce:

$$\bar{d}.\ \frac{x}{y}\neq\frac{x'}{y'}.$$

Proof: 1. Since z' is not parallel to z, construct z'', through B, parallel to z, intersecting the prolongation of AC in C'' (the proof will not be affected if z'' intersects AC between C' and C).

2. In the newly formed triangle ABC'' since $z\|z''$

$$\frac{x}{y}=\frac{x'}{C'C''}.$$

3. But since $y'\neq C'C''$,

$$\therefore\ \frac{x}{y}\neq\frac{x'}{y'}.$$

Since we have shown that abc$\bar{e}$ imply $\bar{d}$ we have thereby shown that its equivalent proposition,

abcd imply e

is also true.

It should be noticed that the appellation

[10] For a fuller treatment of this topic, see The Mathematics Teacher, 31: 216–222, May 1938, or the reprint in book form, pp. 41–47.

[11] See pp. 232–233.

"indirect proof" fits this technique, too, since we have shown the original theorem to be true *indirectly*, through the intermediation of another theorem which is equivalent to it.

The *third* method of indirect proof depends on the definition of contradictory statements which states in part that "if one of two contradictory statements is false, the other must be true."

Let us apply this *approach* to the schematized theorem

$$h_1h_2h_3h_4 \text{ imply } c.$$

If for some reason it is not possible, or not desirable, to show that c follows from h_1, h_2, h_3, and h_4, we can exhibit its truth by showing the falsity of its contradictory.

The contradictory of the above implicative is that $h_1h_2h_3h_4$ do *not* imply c. But this contradictory is of little value since there is no technique available for showing that a set of statements does not imply another statement.

But fortunately there is another form that the contradictory of an implicative proposition can take, and that is, as pointed out above, the assertion that $h_1h_2h_3h_4$ is consistent with $\bar{c}$. Unfortunately there is no deductive technique for showing that a set of statements is consistent with another statement. But this lack of technique is no barrier in *this* case, since the purpose of asserting this statement is only to show that it is false; i.e., that

$$h_1h_2h_3h_4 \text{ is } \textit{not} \text{ consistent with } \bar{c},$$

which is the same as showing that

$$h_1h_2h_3h_4 \text{ is inconsistent with } \bar{c}.$$

The technique for showing the inconsistency of a set has already been pointed out above.

Specifically, this approach will work out as follows, using again the same illustration as on pp. 232–233.

We shall show that abcd imply e by showing the falsity of its contradictory, "abcd are consistent with $\bar{e}$," by showing that abcd and $\bar{e}$ are inconsistent. In carrying out the last step we shall follow, of course, the same procedure as on p. 233. The conclusion will, however, be different. In this case we shall say as follows:

> Since it has been shown that abcd and $\bar{e}$ are inconsistent it follows that the statement abcd and $\bar{e}$ are consistent is false. But since the latter statement is the contradictory of the original theorem,
>
> abcd imply e
>
> and since it has been shown to be false, the original theorem must therefore, by definition of contradictories, be necessarily true.

The Place of These Three Methods of Indirect Proof in Secondary Mathematics

The three methods of Indirect Proof exhibited here are, then—in inverse order—as follows:

I. The method of contradiction
II. The method of contraposition
III. The method of inconsistency.

Although the methods are psychologically different, they are, of course, logically equivalent. The writer never attempted to use the first method mentioned—the method of contradiction—in his classes of plane or solid geometry, for the following reasons:

1. The method is too subtle, and too complicated for the average high school student to warrant all the time and energy that would have to be expended for an adequate presentation of its theory and application.

2. Moreover—and this is the more important reason—the method does not give the logical insight that is necessary for an understanding of the patterns of indirect proof that the student is likely to meet either in the elementary mathematics of the high school, or in the advanced mathematics of the college and university.

3. There is little resemblance between the logical pattern of indirect proof by

contradiction as outlined here and similar modes of reasoning in everyday life.

II. The second method—by contraposition—has a long and honored history.[12] Hauber (1829), De Morgan (1849), The Association for the Improvement of Geometrical Teaching (1875), Wilson (1881), Halsted (1885), McMahon (1903), Smith, E. R. (1909) have advocated its use as *a substitute* or as an alternate for the more familiar and traditional indirect proof in geometry. Even a hasty perusal of textbooks of geometry published in the last two generations will convince the reader that this tendency gained few followers.

In 1938 the writer attempted to revive interest in this method by extending its range of applicability to include theorems of *any* number of conditions in the hypothesis and in the conclusion. While a few recent textbooks have included the treatment of the concept of the contrapositives of a proposition together with its converses and inverses, none found it advisable to advocate its adoption as a substitute for the indirect method of proof, or at least as an alternative method to it. A small number of teachers throughout the country have adopted the use of the contrapositive approach to indirect proof. But admittedly, they, too, have not been successful in convincing their colleagues to adopt this substitute for indirect proof. The reasons for the failure of the spread of this method which is pedagogically simpler than the traditional one, are threefold:

1. The difficulty for any new idea to get a start.
2. The dependence of the teacher of mathematics upon the textbook.
3. The lack of similarity between the contrapositive method and the type of indirect proof that the teachers have been accustomed to.

III. The method of inconsistency that is advocated here has been used by the writer for the last five years in his classes in plane and solid geometry.

[12] For more details, see op. cit., THE MATHEMATICS TEACHER, May 1938.

For obvious reasons his own predisposition in its favor rules him out as an unbiased judge in this matter. He should therefore like to urge the readers of this article to try the method in their own classes in geometry, and would greatly appreciate any comments both favorable and unfavorable they should care to send him.

AN ANALYSIS AND CRITIQUE OF A FEW METHODS OF INDIRECT PROOF

I. *The indirect proofs in Euclid*

Until the early years of this century very few geometry books took the trouble to give the logical basis underlying the method of indirect proof. In the main, the proofs followed closely the one found in Euclid's Elements. Let us examine one that is found in many textbooks:[13]

If a straight line falling on two other straight lines makes the alternate angles equal to one other, the two straight lines shall be parallel to one another.

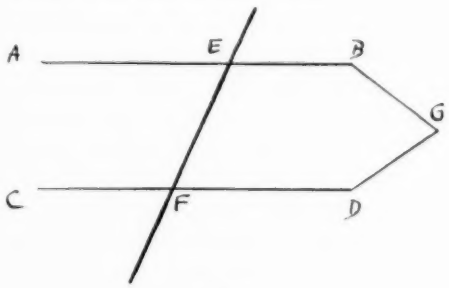

FIG. 4

Let the straight line EF which falls on the two straight lines AB and CD make the alternate angles AEF, EFD equal to one another: AB shall be parallel to CD. For, if not, AB, CD being produced will meet either towards B, D or towards A, C.

Let them be produced and meet towards B, D at the point G.

Therefore GEF is a triangle and its

[13] The proof given here is taken from Todhunter, I., "The Elements of Euclid" 1874, p. 31. This edition has been reprinted as no. 891 of Everyman's Library by E. P. Dutton & Co. in 1933, with an Introduction by Sir Thomas L. Heath.

exterior angle AEF is greater than the interior opposite angle EFG (I. 16). But the angle AEF is also equal to the angle EFG; (Hyp.) which is impossible.

Therefore AB and CD being produced do not meet toward B, D. In the same manner, it may be shown that they do not meet toward A, C. But those straight lines which being produced ever so far both ways do not meet, are parallel (Def. 35). Therefore, AB is parallel to CD.

Wherefore, *if a straight line*, etc. Q.E.D.

Let us examine the structure of this proof. The object of this theorem—like that of every theorem in mathematics—is to show that the conclusion is deducible from the hypothesis and from some other relevant theorems, postulates and definitions that have preceded this theorem. This objective has certainly not been achieved in this proof. What has been done was to show that from the combination of the denial of the conclusion and the use of the statements in the hypothesis it is possible to deduce the false statement that one angle AEF is both equal to and greater than a second angle EFG. This is equivalent to the statement that an angle is equal to and greater than itself. In the terminology of the analysis advocated in this paper, what is shown in this proof is the inconsistency of the denial of the conclusion and the statements in the hypothesis. But unless an explicit statement is made somewhere in the proof of the *logical* equivalence of an inconsistent set and the original implicative theorem, the latter has not been demonstrated. For, the object is not to prove the isolated fact that AB is parallel to CD, but to show that it follows from the given conditions.

Some textbooks contain a variant of the above demonstration. The proof is exactly the same until the completion of the statement,

> "Therefore GEF is a triangle, and its exterior angle AEF is greater than the interior opposite angle EFG."

At this point the following statement occurs.

> "Which is contrary to the hypothesis."

It should be observed that in this method of proof the textbook writer is really showing that from the denial of the conclusion together with the use of all *but one* of the conditions in the hypothesis it is possible to deduce the contradictory of the remaining condition of the hypothesis. But this method of proof is no more than an unconscious use of the method of contraposition described above. Again, unless the equivalence between a contrapositive and the original theorem is somewhere stated in the proof, the proof is psychologically not convincing and logically not complete. Without this statement of equivalence, neither the traditional conclusion—"*Wherefore, if a straight line*, etc." nor the comforting, almost triumphant "Q.E.D.," is logically justifiable.

II. *An early attempt at rationalizing the indirect proof*

One of the early attempts by textbook writers at establishing the logical basis of indirect proof was made by Alan Sanders.[14] In anticipation of the indirect proof of Theorem III,

> *At a given point in a line only one perpendicular can be erected to that line,*

the author supplies the following detailed explanation:

> "The next proposition is an example of what is called the *indirect proof*.
>
> The reasoning is based on the following Principle:
>
> *If the direct consequences of a certain supposition are false, the supposition itself is false.*
>
> To prove a theorem by this plan, the following steps are necessary:
>
> 1. The theorem is supposed to be untrue.

[14] Sanders, Alan, "Elements of Plane and Solid Geometry, New York, American Book Co., 1901, 1903, p. 18.

2. The consequences of this supposition are shown to be false.
3. Then, by the above Principle, the supposition (that the theorem is untrue) is false.
4. The theorem is therefore true."

When the actual indirect proofs of a few theorems are examined, the following points will be observed:

1. The author believes that he is supposing the theorem to be untrue, but since he is only contradicting the conclusion, he is only getting the contrary of the original theorem and not its contradictory.
2. In no theorem does the author really attempt to show the falsity of the contrary theorem that he purports to do.
3. Even if he did show that the contrary is false, the original theorem would not thereby be proved true indirectly.
4. In the plan quoted above the author says "that the consequences of this supposition are shown to be false."

In actual practice he obtains a false consequence by either one of two ways:

a. by combining the denial of the conclusion with the truth of *all* the conditions of the hypothesis. In this case he is unconsciously using the concept of an inconsistent set.

b. by combining the denial of the conclusion with the truth of *all but one* of the conditions in the hypothesis. The false consequence obtained is the denial of the remaining condition in the hypothesis. In the latter case the author is really using the Law of Contraposition.

III. *A later attempt at rationalizing the indirect proof*

The influence of the above analysis by Sanders was not strong. Very few of the textbooks published in the first three decades of this century seem to have adopted his approach.

Many of the books published since 1930 have adopted a pattern of the indirect proof, which resembles the one analyzed and advocated by C. B. Upton.[15]

His analysis is deeper and more thorough than Sanders'; and is based on the following three principles:

Principle I. The Law of Excluded Middle. *A thing must either be or not be.*

Principle II. *If one of two contradictory propositions is proved to be true, it immediately follows that the other must be false; similarly, if one of them is proved to be false, it immediately follows that the other must be true.*

Principle III. *If the conclusion of a correct process of reasoning be false, then the premises from which it necessarily follows must also be false.*

Nearly all the comments that were made above on Sanders' analysis apply with equal force here too, since Principle III is essentially the same as the principle enunciated by Sanders. To be completely true, principle III should be amended to include the words in brackets:

> If the conclusion of a correct process of reasoning be false, then [at least one of] the premises from which it necessarily follows, must also be false.

One more observation on the above approach must be made. Everyone of the indirect proofs that is worked out in detail begins with the sentence "One of these propositions must be true" followed by the original conclusion and the contradictory of the original conclusion. In this analysis, just as in the one by Sanders, the conclusion seems to become the focus of attention, while the conditions in the hypothesis remain only in the vague background, with the result that the necessary connection between the two is lost sight of. The writer believes, however, that if the law of contradiction is to be used explicitly, the alternatives should be formulated in the following way so as to include references to

[15] Upton, C. B., "The Use of Indirect Proof in Geometry and in Life," Chap. X, pp. 102–133. "The Teaching of Geometry," Fifth Yearbook, National Council of Teachers of Mathematics, 1930, Bureau of Publications, Teachers College, Columbia University.

the *connection* between the hypothesis and the conclusion.

One of these propositions must be true:

(1) *the hypothesis implies the conclusion,*

(2) *or the hypothesis does not imply the conclusion.*

Showing alternative (2) to be false would be tantamount to showing alternative (1) true. This proposed approach is, of course, the method of contradiction analyzed above on p. 235.

Perhaps the best way of illustrating the misleading tendencies that are inherent in discussing the contradictory of the conclusion *without* reference to the conditions in the hypothesis is to give a few examples of such analyses.

Imagine a student who is wondering whether the following statement is true or false:

If a line that intersects two sides of a triangle cuts off a small triangle that is similar to the larger triangle, it is a parallel to the third side.

Not being able to deduce the conclusion directly, he formulates the following attack in accordance with the analysis that is under discussion.

One of these statements must be true

(a) *the line is either parallel to the third side*

(b) *or the line is not parallel to the third side.*

At first sight it might seem as if one, *and only one* of these statements can be true. But a little reflection will show the reader that *both* possibilities are consistent with the hypothesis, and *neither* is deducible from it. For in any triangle ABC it is always possible to draw a line $C'B'$ (technically called an *anti-parallel*) which will intersect AB and AC in C' and B' respectively such that angle $AC'B' =$ angle ACB. This line—$C'B'$—will in general not be parallel to BC (except where $AB = AC$), but $\triangle AC'B'$ will always be similar to $\triangle ABC$.

Similar exercises can be created almost at will where the given conclusion and its contradictory can both be true but neither is deducible from the hypothesis. Just one more example will suffice.

Imagine the same student wondering whether the following theorem is true:

Given: $\triangle ABC$ is equal in area to $\triangle A'B'C'$

$AB = A'B'$

$BC = B'C'$

to deduce $\angle B = \angle B'$

and again he will say to himself:

One of the two statements must be true, either $\angle B = \angle B'$ or $\angle B \neq \angle B'$.

Again this analysis will mislead him, for the second conclusion, too, may be true as in the case of $\angle B$ being the supplement of $\angle B'$ without being equal to it.

The error here too is due to the *dissociation* of the conclusion from the premises, and to dealing with the former independently of the latter.

Summary

The fundamental flaws pointed out above in the various attempts at rationalizing the indirect proof have convinced the writer of the need of another approach. He believes that the method of inconsistency advocated here is logically sound and psychologically convincing. He hopes that his readers will give it a trial.

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◆ THE ART OF TEACHING ◆

Centroids

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In geometry classes we occasionally meet with the question: "What is geometry good for?" Usually the description of practical applications will provide the necessary answer. But in some cases the same question if it had been more accurately formulated, would read: "Can you convey to me experiences showing that geometry has a meaning and reality beyond man-made definitions and theorems?" Then the answer could be given somewhat in the line of the following example. One would say to the student: "Let me show you something. Here is a piece of cardboard in shape of a triangle. Try to balance it on the eraser-end of your pencil." The student will make a few attempts and finally succeed in finding the point where the cardboard has to be supported in order to uphold itself in equilibrium. Then one will show that the same point can be obtained without any experimentation through geometric construction. It is the point of intersection of the medians of the triangle (Figure 1).

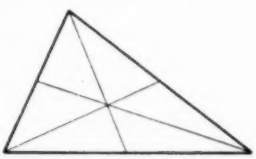

Fig. 1. Centroid of a triangle.

A study of centroids is especially fit to build the bridge between geometric constructions and facts related to natural sciences. The position of a centroid is independent of the circumstance whether one measured in inches or centimeters and of whatever terms or definitions had been used in support of the constructions.

From the centroid of a triangle one can proceed to the centroid of a quadrilateral. A diagonal divides a quadrilateral into two triangles. For each of the triangles the centroid can be obtained as the point of intersection of two medians. An edge supporting both centroids balances the quadrilateral; the straight line joining the centroids is a locus for the centroid of the quadrilateral (see Figure 2, left diagram). The second diagonal also divides a quadrilateral into two triangles which are different from the first ones. The straight line joining their centroids is again a locus for the centroid of the quadrilateral (see Figure 2, right diagram). Therefore the centroid of the quadrilateral is found as the point of intersection of the two loci (see Figure 2, third diagram). By means of cutting the given quadrilateral out of a piece of cardboard the position of the centroid can be checked experimentally.

The 8 medians through which the centroid of the quadrilateral has been obtained form a stellar octagon which is inscribed in the quadrilateral. Figure 3 shows a stellar octagon inscribed in a circle. The circle is divided into 8 equal parts and every point of division is joined with the third one following it in either way around the circle. Figure 4 shows a stellar octagon inscribed in a square. The 8 points which are used to construct the stellar octagon are the four vertices and the four middle points of the sides of the square. Each point is joined with the third one following it in either way around the square. In Figure 5 the construction of

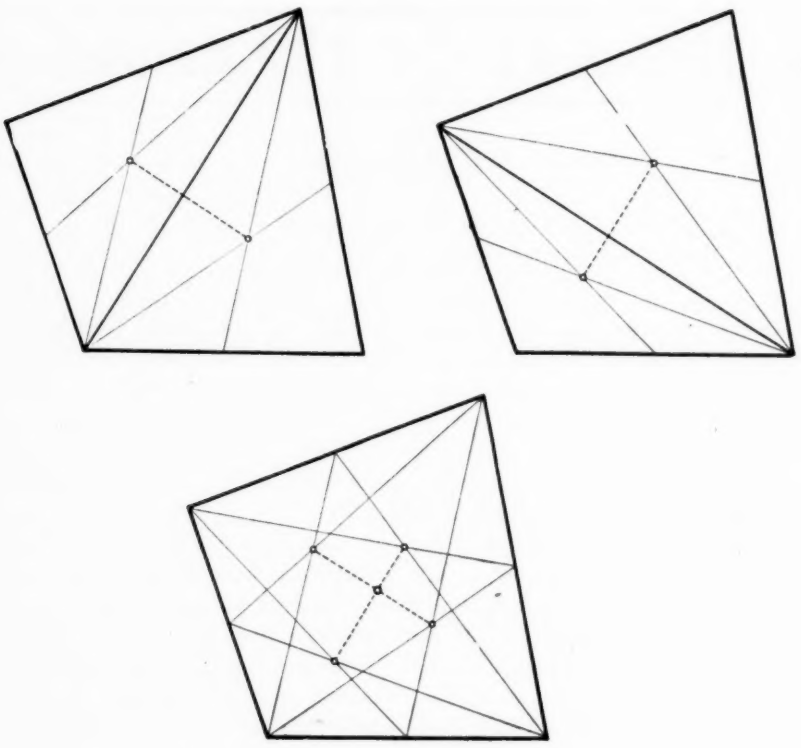

FIG. 2. Centroid of a quadrilateral.

an inscribed stellar octagon is repeated once more for an irregular quadrilateral, using again its vertices and the middle points of its sides. The dotted lines connecting the centroids of the partial triangles intersect in the centroid of the quadrilateral.

A short cut in the construction of the centroid of a quadrilateral can be achieved through the method shown in Figure 6. The given quadrilateral ABDE is divided through the diagonal BE into two triangles (ΔABE and ΔBDE). In each triangle two medians are drawn and their points of intersection C_1 and C_2 are the centroids. The straight line $C_1 C_2$ connecting the centroids contains the centroid of the quadrilateral. Would the two triangles have been of the same area their common centroid C would be half way between C_1 and C_2. But as soon as one triangle is larger than the other C lies closer to the centroid of the larger triangle. Following the law of moments the ratio of the distance of C from C_1 and C_2 is the reciprocal values of the ratio of the areas A_1 and A_2 of the respective triangles:

$$\frac{CC_1}{CC_2} = \frac{A_2}{A_1}.$$

The areas are proportional to the altitudes of these triangles drawn perpendicularly to their common base. In Figure 6 the altitude a_1 is transferred to the perpendicular to C_1C_2 erected in C_2 and the altitude a_2 to the perpendicular to C_1C_2 erected in C_1. The inclined line joining the endpoints of the perpendiculars intersects C_1C_2 in C. Through the law of similar triangles we get

$$\frac{CC_1}{CC_2} = \frac{a_2}{a_1} = \frac{A_2}{A_1}.$$

Therefore C is the common centroid of the triangles and the centroid of the quadrilateral.

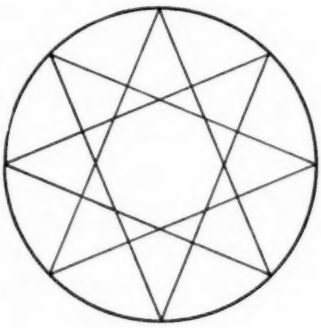

FIG. 3. Stellar octagon inscribed in a circle.

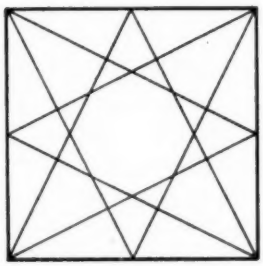

Fig. 4. Stellar octagon inscribed in a square.

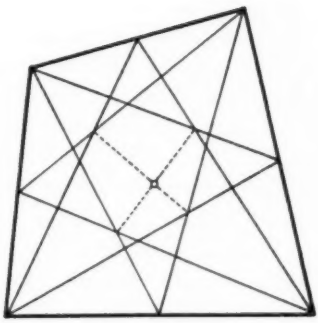

FIG. 5. Stellar octagon inscribed in a quadrilateral. Centroid.

In Figure 7 the procedure is simplified to merely transferring the distance d between C_2 and the point of intersection S of C_1C_2 with the diagonal BE: $C_2S = C_1C$. This construction is based on the fact that C_2S and C_1S are proportional to one-third of a_1 and a_2 respectively, therefore also to a_1 and a_2 themselves and to the areas of the two triangles.

These constructions can also be applied to find the centroid of polygons. The procedure is shown in Figure 8 and Figure 9 for an irregular hexagon: $ABDEFG$. In Figure 8 the diagonals from one vertex A of the hexagon are drawn: AD, AE, AF. These diagonals divide the hexagon into 4 triangles. In each triangle the centroid is obtained through the medians. From the centroids C_1 and C_2 the combined centroid C_I is then constructed through the method of Figure 7, and so also the combined centroid C_{II} between C_3 and C_4. Finally C_I and C_{II} are combined to the common centroid C. This has been carried out in Figure 9 following again the principle of Figure 7. The only difference between the constructions of Figure 9 and Figure 7 lies in the fact that Figure 7 deals with 2 triangles on a common base whose altitudes have the ratio of their areas whereas Figure 9 deals with 2 quadrilaterals. The quadrilaterals can be transformed into triangles with the common base AE: From the quadrilateral $ABDE$ the triangle BDE is cut off through the diagonal BE and is replaced by the triangle EBH which is of the same area because the line DH is drawn parallel to BE. Following the same

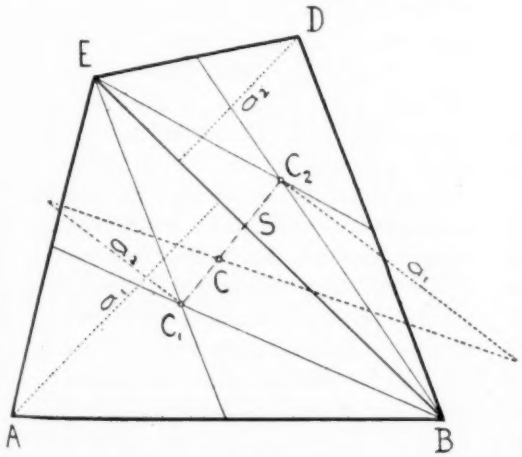

FIG. 6

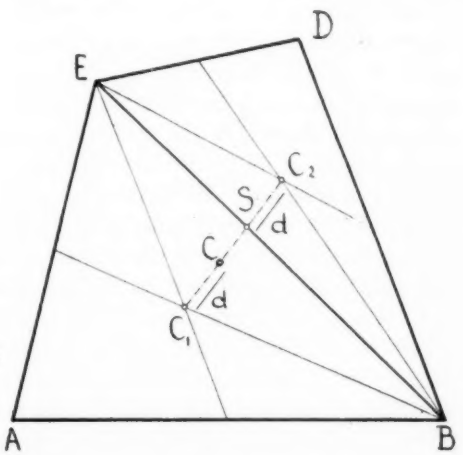

FIG. 7

Further constructions of the centroid of a quadrilateral.

construction the triangle EGF is cut off from the quadrilateral $AEFG$ and replaced by the triangle GEK of the same area (FK is parallel to EG). After the quadrilaterals have been replaced by triangles their altitudes a_1 and a_2 perpendicular to the common base can be used in the same way as in Figure 7 to find the centroid C.

The results of all the constructions of centroids can be verified through experiments. A piece of cardboard cut out in shape of the given polygon will be in balance when supported at the centroid.

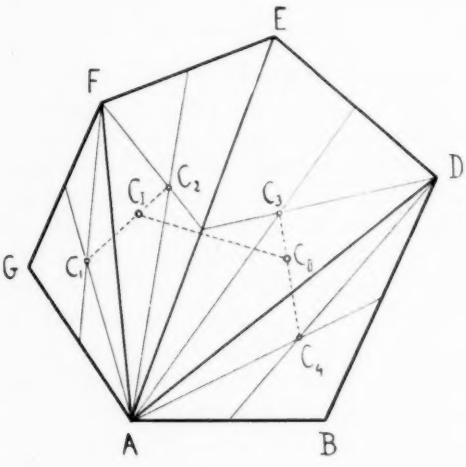

FIG. 8

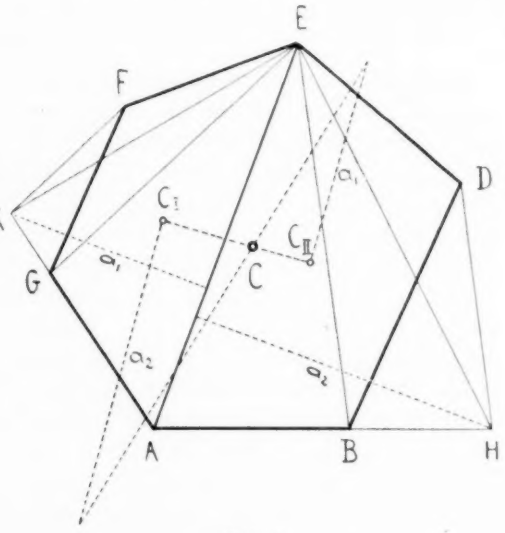

FIG. 9

Construction of the centroid of a hexagon.

There are also various other ways to find centroids experimentally. Take a piece of cardboard and suspend it on a thread by making a cut with a razor blade into the cardboard at any point along its edge and pulling a thread through it. Friction will uphold the cardboard on the thread. When the cardboard is suspended the centroid lies exactly vertically under the point of suspension. It is therefore located along a vertical line which can be drawn on the cardboard from the point of suspension perpendicularly downward. The latter can be drawn with the aid of a pendulum consisting of a small weight, for instance an eraser, hanging on a thread. By suspending the cardboard successively at two different points two lines are obtained whose point of intersection is the centroid. All additional lines derived from further points of suspension also concur in the centroid.

A second way to obtain the centroid of an area by experiment is to hold the cardboard in a vertical plane and balance it on one point of its lower edge. After the position in which the cardboard will balance on this point has been determined a vertical line drawn upward from the point of support on the cardboard is a locus of the centroid. A pendulum can again be used to obtain the exact position of this line. Two such lines drawn from two different points of support intersect in the centroid. Further lines obtained in the same way concur in the centroid. This method works best with larger pieces of cardboard; smaller ones are more handy to be suspended than supported.

A third way consists in balancing the cardboard in horizontal position on a straight edge, for instance on the edge of a ruler. Any straight line drawn across the cardboard along which it will balance is a locus of the centroid. Two different experiments yield two different straight lines and their point of intersection is the

centroid. All other straight lines along which the cardboard can be balanced also concur in the centroid.

A fourth method proceeds along the way which is followed to derive the formula for the coordinates of the centroid in the calculus. It uses two rulers, the cardboard and a duplicate of the cardboard. One ruler is held in horizontal position and is balanced on the edge of the second ruler (see Figure 10). One of the cardboards is placed in horizontal position on one side of the ruler and the duplicate is hung from a thread on the other side of the ruler. By placing this thread closer to or further away from the fulcrum its position can be determined in which the two cardboards will be in equilibrium. The centroid of the horizontal piece of cardboard is located at the same distance d from the balancing edge as the thread on which the duplicate is suspended. Therefore a straight line drawn on the cardboard at a distance d from the balancing edge is a locus of the centroid. By changing the position of the cardboard which lies horizontally on the ruler and repeating the experiment a second locus for the centroid will be obtained. The point of intersection of the two loci is the centroid.

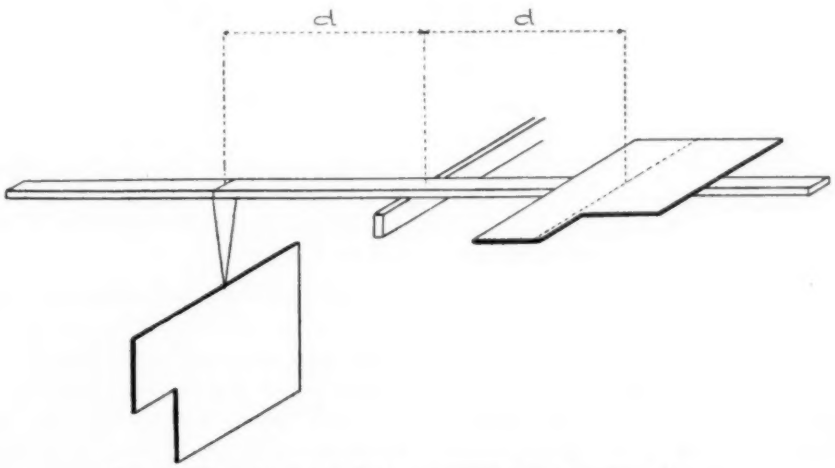

FIG. 10. Experiment to find the centroid of a cardboard-area.

An experiment which demonstrates the position of centroids for areas which have an axis of symmetry is the following: After the axis of symmetry has been marked on the cardboard one aims to suspend it on a thread in such a manner that the axis will stay in horizontal position. As every axis of symmetry is in itself a locus for the centroid the latter is its point perpendicularly under the point of suspension. The same method can be applied to solids, for instance to wooden models suspended on threads which are held to the solids by strips of scotch tape. Figure 11 shows the positions of centroids for areas and solids. The centroid divides the axis in certain ratios, as $\frac{1}{2}$ for a rectangle, prism or cylinder, $\frac{1}{3}$ for a triangle or a paraboloid, $\frac{1}{4}$ for a pyramid or a cone, $\frac{2}{5}$ for a parabola segment and $\frac{3}{8}$ for a hemisphere. For the semi-circle the constant π appears in the ratio which is $4/3\pi$. Those ratios are obtained through the calculus.

Another experiment (Figure 12) shows the balancing of a line. A wire is bent in a broken line composed of six straight line segments. The wire will stay in equilibrium with the first and last segment in horizontal position when it is suspended or supported at the point C. The figure also shows the construction of point C by the methods of graphic statics using a "force and equilibrium polygon." At the middle point of each segment are

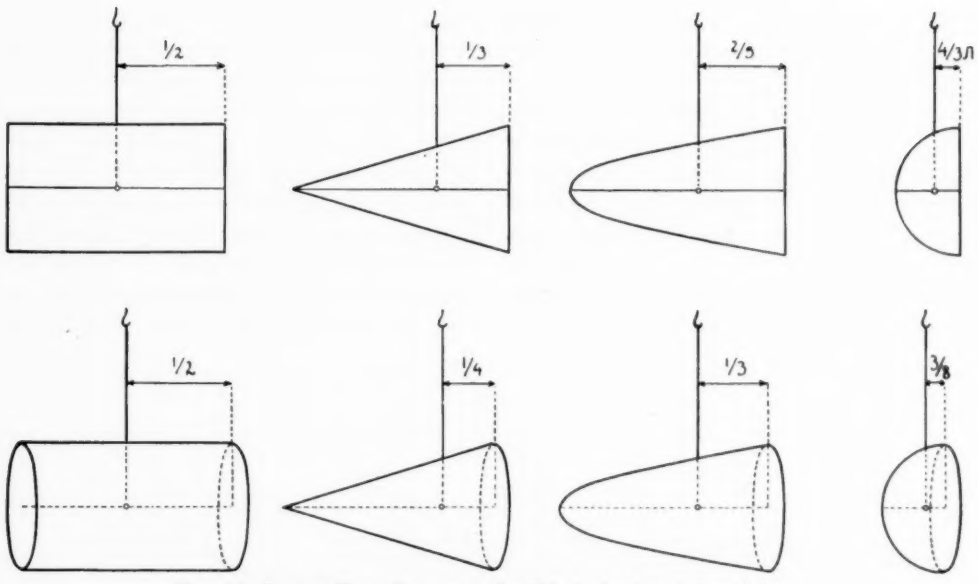

FIG. 11. Suspending of areas and solids in horizontal position. Ratios of the positions of the centroids.

drawn arrows representing the forces of gravity. The lengths of the arrows are proportional to the lengths of the segments. In Figure 12 each arrow has half the length of a segment. The same lengths are transferred to the force polygon (right diagram of Figure 12). The end points of all the segments are connected with a point P that may be chosen at will. From these lines the equilibrium polygon is obtained, which is drawn between the vertical lines dropped from the six arrows. Between the vertical lines of 1 and 2 a parallel to the line in the force polygon

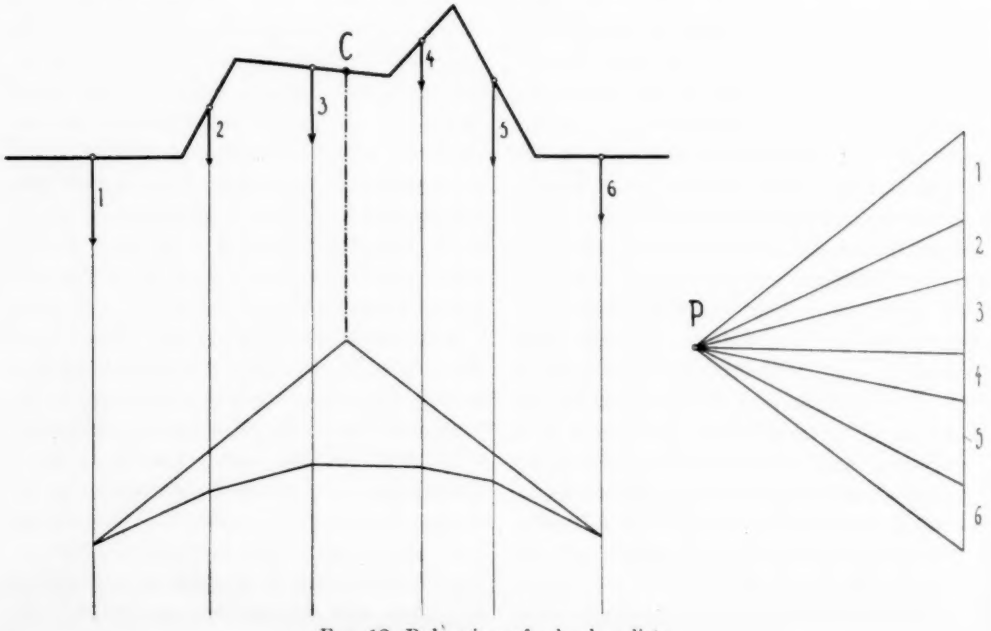

FIG. 12. Balancing of a broken line.

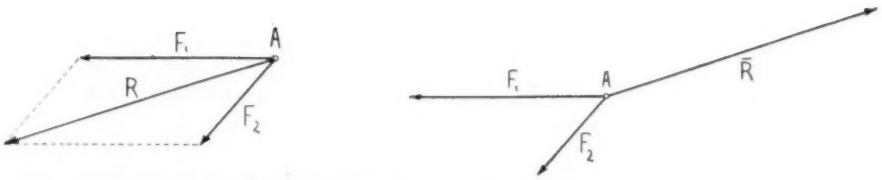

FIG. 13. Parallelogram of forces combining two concurrent forces.

connecting P with the point between the segments 1 and 2 is drawn. Then a second parallel line is drawn from the end of the first one between the verticals 2 and 3, parallel to the line connecting P with the point between the segments 2 and 3. So one continues until the last vertical is reached. Then from the end point of the last parallel a line is drawn parallel to the connection of P with the end of segment 6 and from the beginning of the equilibrium polygon a line parallel to the connecting line from P to the top of segment 1. From the point of intersection of these last two lines the vertical line which is shown in the figure in dots and dashes is drawn upwards and determines C. In this procedure three kinds of constructions are contained in a condensed form.

The first is the construction of a parallelogram of forces which combines two forces (F_I and F_2 in Figure 13) acting on an object at a point A. The resultant force R is obtained, both in its size and direction, as the diagonal of a parallelogram of forces. A force $\overline{R}$ of the same strength but acting on A in the apposite direction balances the forces F_I and F_2 and keeps the object in equilibrium (Figure 13, right diagram).

A second construction combines two forces which act on two different points (Force F_1 acting on point A and Force F_2 on point B which both lie on a rectangular board drawn in Figure 14). The construction proceeds by extending the arrows of the given forces backwards until they intersect. At the point of intersection C the parallelogram of forces is drawn which yields the resultant R. A force $\overline{R}$ equal in strength to the resultant R but acting in the opposite direction on any point C along the diagonal of the parallelogram of forces will keep the board in equilibrium (Figure 14, right diagram).

The construction of Figure 14 can be carried out for any two forces except for parallel forces. But cases of parallel forces occur especially frequently as they include the forces of gravity. Parallel forces can be handled through the trick of adding an arbitrarily chosen pair of equal opposite forces. The procedure is shown in Figure 15, left diagram. The given forces are F_1 acting on point A and F_2 acting on point B. The additional pair of forces consists of G_1 and G_2. First F_1 and G_1 are combined in a parallelogram of forces and their resultant is R_1. Then R_1 and F_2 are combined as non-concurrent forces and their resultant is R_2. Finally R_2 and G_2 are combined as non-concurrent forces and their resultant R is directed perpendicularly downwards, its length being equal to the sum of the lengths of the arrows F_1 and F_2.

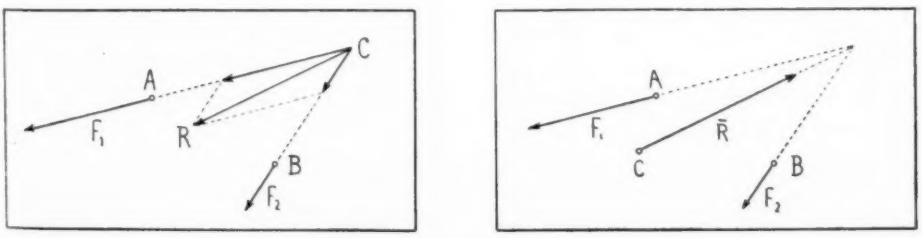

FIG. 14. Parallelogram of forces for two non-concurrent forces.

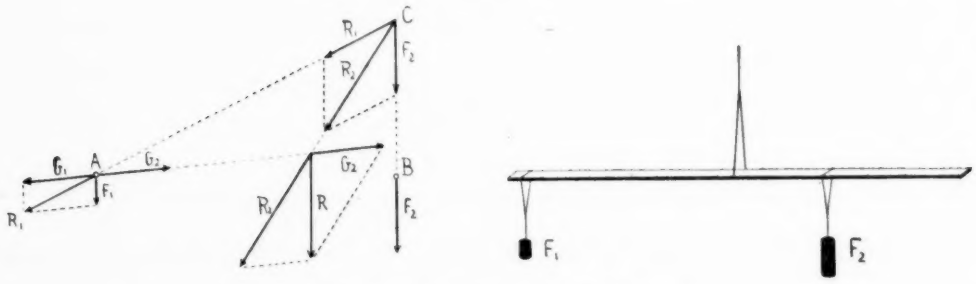

FIG. 15. Resultant of two parallel forces.

A force $\overline{R}$ of the same strength as R but with the opposite direction acting on the same point as R restores the equilibrium. In the right diagram the forces F_1 and F_2 are interpreted as weights suspended on a ruler. The ruler stays in equilibrium when supported at a point with the same distances from the given forces as R.

Repeated application of this construction solves the problem to find the centroid of the wire in Figures 12 and 16. The forces of gravity which apply to the 6 sections of the wire are denoted simply as 1, 2, 3, 4, 5 and 6. The additional pair of forces is G_1 and G_2. First 1 and G_1 are combined in a parallelogram of forces which yields a resultant. Then this resultant is further combined with force 2 and so forth until one arrives at the last resultant which is finally combined with G_2. Thus the position of the ultimate resultant is reached. On the right hand of this diagram the force polygon of Figure 12 is repeated. In it the lengths of the segments 1 to 6 equal those of the arrows 1 to 6. The position of the point P has been so chosen that PA equals G_1. A systematic comparison of the force polygon with the diagram on the left side of Figure 16 will show that the force polygon is but a condensation of this

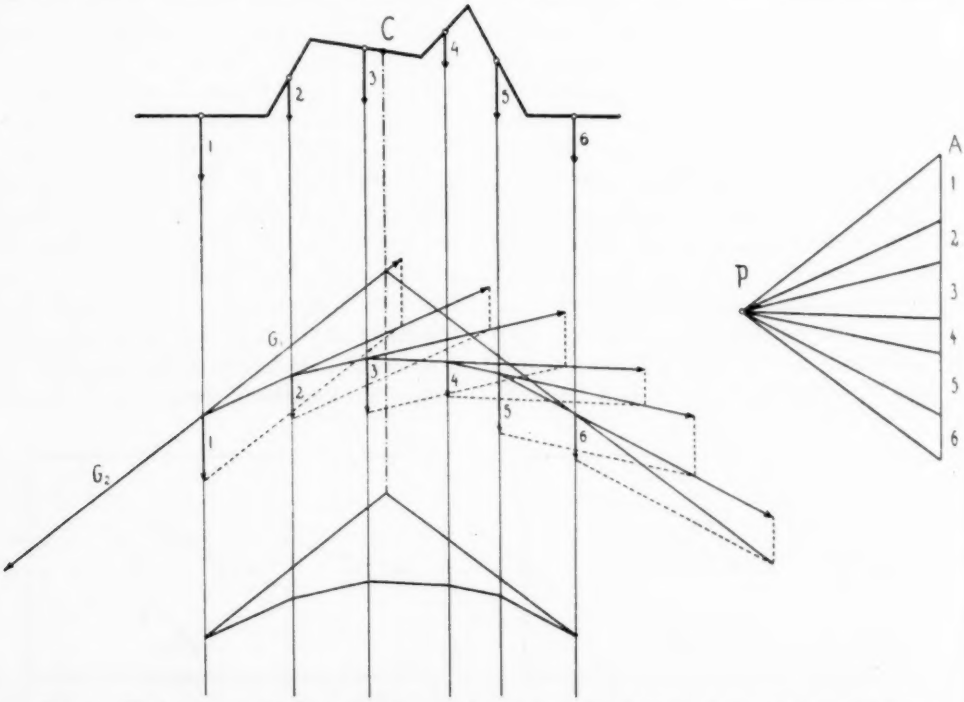

FIG. 16. Derivation of the construction of the centroid by force and equilibrium polygons.

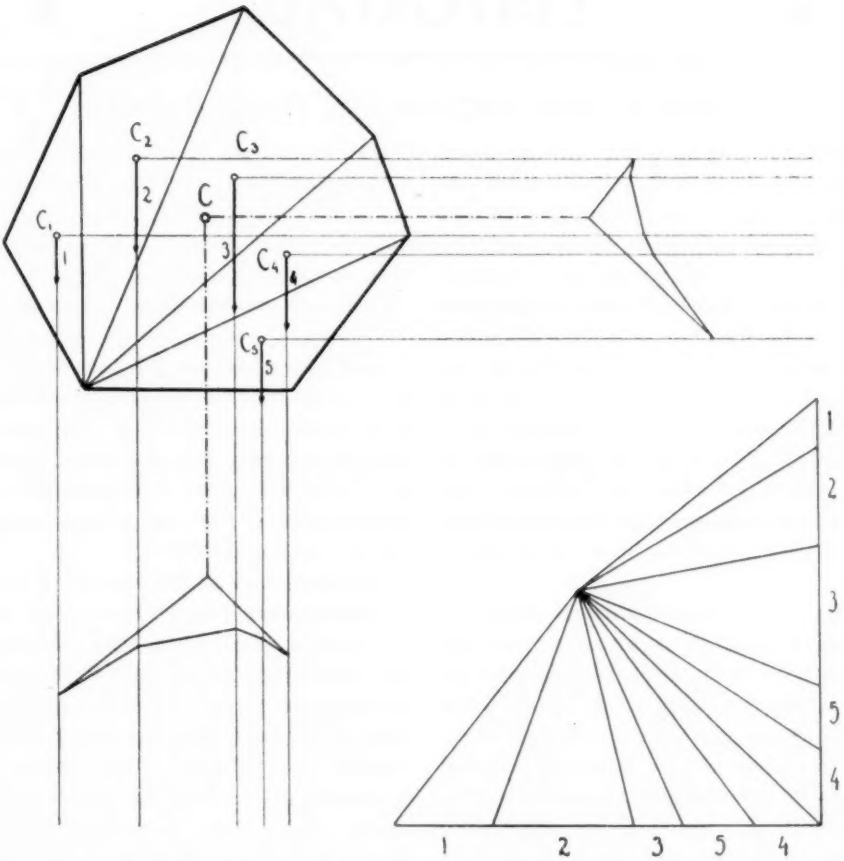

FIG. 17. Construction of the centroid of an area through force and equilibrium polygons.

diagram and is composed of triangles which are congruent to those which appear in the parallelograms of forces. Thus the construction of the force and equilibrium polygons of Figures 12 and 16 derives itself from the basic constructions of the Figures 13 to 15.

Finally the construction of force and equilibrium polygons can be applied to an irregular polygon, as in Figure 17. The diagonals drawn from one vertex of the polygon divide it into triangles. For each triangle the centroid has been obtained (through medians not shown in the diagram). From the centroids of the triangles arrows are drawn representing the forces of gravity whose lengths are proportional to the areas of the triangles. From the lengths of these arrows the force polygon has been set up and through it the equilibrium polygon which is a locus of the centroid (See dots and dashes). Imagining then that the given area with the diagram be turned 90° so that the parallel lines which are drawn from the centroids of the triangles to the right come into perpendicular position, the construction of the force and equilibrium polygon can be repeated once more. Thus a second line shown in dots and dashes is obtained which is also a locus for the centroid of the area. The point of intersection of the 2 loci is the final centroid C. The same construction can be carried out without an excessive amount of lines to find the centroid of any given polygon. Its results will be the same whatever vertex may have been chosen to draw the diagonals across the polygon or whatever decision has been made to divide the area into triangles. The centroid thus obtained can be checked by experiment.

EDITORIAL

TEACHER EDUCATION IN THE YEARS AHEAD

SOMEONE has said, "Teachers are born, not made," but surely there is something in training. Those of us who have watched the progress of a student teacher from the first time he is given a chance to teach a class until he either turns out to be a good teacher or he does not, will have difficulty in subscribing to the above statement. To be sure, if he does not develop into a good teacher, someone may say, "You see, it is not in him." But obviously that is not all of the story. How can we be sure that under proper training and supervision the result might not have been quite different?

Student teaching conditions have existed under which those who were learning to teach have been subjected to great indignities. On the other hand, who has not seen the wholesome influence and effect on many students whose supervisors used methods that developed them into artistic teachers. If one can pick the supervising teacher, he can be confident of fairly certain satisfactory results. Of course there will always be a few misfits, even under the best conditions, but the number of such people should be constantly reduced in the future.

In the past, the teaching profession has not attracted a large enough number of the proper types of people. It has sometimes been so bad as to put those of us who chose to go into teaching in a bad light. We were looked down upon. You will remember the familiar remark, "Those who can, do, and those who can't, teach." Well now, is that true? But be that as it may, the best has been none too good, and some of the good might have been better. Why? How?

The American people have been getting from the schools, and will continue to get, just about what they pay for, no more, no less. To date they have been unwilling to pay much of a price. As a nation we spend more on chewing gum, cosmetics, and liquor to say nothing of war, than we do on education. What a significant implication that comes to mind concerning American intelligence. The American people today probably have a much higher opinion of the teacher than they did when World War II began, because of the excellent work done by the teachers in connection with the rationing program. But should it take a war to awaken the nation to the latent power and influence of the classroom teacher upon the lives and destiny of children?

We have had an oversupply of teachers in some places, but we have never had an oversupply of good teachers, and we won't get them until salaries are so increased as to attract men and women of high calibre who in the past have not been willing, no matter how much they might have wished to do it, to adopt teaching as a life work. Then, too, because of the high mortality of women teachers due to marriage, we must attract more men who will adopt teaching as a career. Once we can do this, how can such men and women best be trained? We hear a great deal nowadays about this question. Whether preservice teachers are trained better in campus or cooperating schools is open to debate. We cannot go into the merits of either plan here, but the question is vital. Then, too, standards for teaching should be raised. With adequate salaries no one should be permitted to teach mathematics, even in the junior high school, who has not had training in the calculus.

Then there is the question of inservice training. Some people seem to think that once they get a teaching position, all further need for training should be forgotten. How fatal is such a notion?

Again we are already being requested by certain schools to train master teachers —teachers who can teach anything under

the sun. We should not fall in too fully and too quickly with this attitude and bring up a generation of teachers who will know "less and less about more and more." We must have a vertical as well as a horizontal view of what we teach and we should have little sympathy with or hope for a teacher of mathematics who says, "We are not teaching mathematics, we are teaching children," or some such foolishness. It is like the "good old days" of the project method, where too often we got "all project and no arithmetic." And, speaking of arithmetic, you can't very well train good teachers of arithmetic in the days ahead with a twelve weeks' course, no matter who gives it. There are two sides to this question, and at heart, like all the other educational panaceas, the idea of an integrated program is sound, but if you carry it to its logical extreme at the present time, it just won't work.

—W.D.R.

◆ IN OTHER PERIODICALS ◆

By NATHAN LAZAR
Midwood High School, Brooklyn 10, New York

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March 1947, Vol. 54, No. 3

1. Salzer, H. E., "The Approximation of Numbers as Sums of Reciprocals," pp. 135–142.
2. DeCicco, John, "Geodesic Perspectivities upon a Sphere," pp. 142–147.
3. Zorn, M. A., "Approximating Sums," pp. 148–151.
4. Evans, G. C., "A Necessary and Sufficient Condition of Wiener," pp. 151–155.
5. Mathematical Notes, pp. 156–158.
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   (b) McBrien, V. O., "Polygons Having a Common Mean"
6. Classroom Notes, pp. 159–162.
   (a) Smiley, M. F., "The Trial Integral Method"
   (b) Farnell, A. B., "On the Solution of Linear Differential Equations"
7. Elementary Problems and Solutions, pp. 163–167.
8. Advanced Problems and Solutions, pp. 168–173.
9. Recent Publications, pp. 174–178.
10. Club and Allied Activities, pp. 179–181.
11. News and Notices, pp. 182–194.

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1. "Program for Joint Meeting of K.A.T.M. and Kansas Section of M.A.A.," p. 36.
2. Sanger, R. G., "A History of Mathematics in America before 1875," pp. 36–38.
3. Neywick, Leslie, "Mathematics in the High School Curriculum," pp. 38–42.
4. Tice, Charles B., "Multi-Sensory Aids in Teaching Mathematics," pp. 43–45.
5. Bartky, A. John, "Guidance in Mathematics," pp. 45–47.
6. "Consumer Mathematics in the Junior High School," pp. 47–48.

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1. Andrews, F. Emerson, "The Society's Annual Award," p. 1.
2. Robert, Jr., Harry C., "The Fibonacci Series," pp. 3–9.
3. Tables of Duodecimal Log Sines, Log Cosines, Log Tangents, and Log Cotangents, pp. 10–11.
4. "Fermat's Factoring Formula," pp. 12–13.
5. "Additions to Duodecimal Bibliography," pp. 13–15.
6. Camp, Kingsland, "Duodecimal Nomenclature—Further Notes," pp. 16–20.
7. DeFrance, Charles Q., "The Expansible Integers," pp. 21–22.
8. Scifres, Eugene M., "Calculating π," pp. 23–24.
9. Lloyd, Mary, "Mathematical Recreations," pp. 25–26.
10. The Mail Bag, pp. 26–29.

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2. "The Place of Mathematics in Secondary (Modern) Schools," pp. 250–271. A discussion in which the following participated: Siddons, A. W.; Swan, F. J.; Caton, C. T. Lear; Watts, H. H.; Smith, E. J. P.; Bushell, W. F.; Grocock, T. A.; Williams, E. M.; Inman, S.; Miss. Meetham; Miller, H. H. F.; Riley, A. W.; Parsons, G. L.; Lowry, H. V.
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4. Gant, P., "A Treatment of the Exponential and Logarithmic Functions," pp. 277–281.
5. Turton, F. J., "A Lesson on π and on $\lim_{\theta \to 0}(\sin \theta/\theta)$, pp. 282–286.
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10. Correspondence, pp 300–305.
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1. Knowler, Lloyd A., "Some Industrial Effects of Quality Control by Statistical Methods," pp. 207–211.
2. Jones, Phillip S., "An Interesting Trend in the Teaching of Algebra," pp. 225–228.
3. Lorentz, Floyd S., "Trisection of the Angle" pp. 255–257.
4. Nyberg, Joseph A., "Notes from a Mathematics Classroom" (Continued), pp. 271–275.
5. Problem Department, pp. 285–291.
6. Book Reviews, pp. 291–297.

An Inquiry—What's Going on in Your School

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The Commission on Post-War Plans of the National Council of Teachers of Mathematics would like to get a picture of what is happening in mathematics in typical schools—large and small. You can help by providing the information requested for your building or better still, for your whole organization if you are part of a city system. A report of trends will, in due time, be published in THE MATHEMATICS TEACHER. Feel free to be altogether frank—the report will in no way refer to you or your school without getting your special permission.

Part III

In many conferences of mathematics teachers there has been much discussion of the "double track." This part of our inquiry deals with the mathematical courses that might be offered on the second track.

Name of School _____
Location of School _____
Name of person reporting _____
Address _____

1. Please fill in the following table:

Grades	Pupils in grade	Pupils enrolled in a mathematics class
7		
8		
9		
10		
11		
12		

2. As concerns refresher courses:
   a. Do you provide any form of refresher courses in the 11th or 12th grade? _____ If so, will you please continue?
   b. Is the course required? _____
   c. Does the course consist chiefly of a review of arithmetic? _____
   d. Is credit given for the course? _____
   e. Does the course last as long as a semester? _____

   This question was proposed by Mary Potter.
3. If you offer general mathematics in grade nine, how would you describe the attitude of:
   a. The teachers toward the course? like dislike indifferent
   b. The pupils toward the course? like dislike indifferent

   (The above question was proposed by H. Vernon Price of Iowa City, Iowa.)
4. Does your school offer
   a. One year of general mathematics paralleling the 9th grade? _____
   b. Two years of general mathematics paralleling the traditional sequential courses in the 9th and 10th grades? _____
   c. If your answer is no, do you think your school should offer two years of general mathematics? _____

5. a. Is your school offering a course in shop mathematics? _____
   b. If yes, then is the course in shop mathematics taught by someone in the mathematics department or by the shop teacher? _____
6. a. Have you read a pamphlet entitled "The Role of Mathematics in Consumer Education" prepared by the Consumer Education Study, 1201 16th St., N.W., Washington 6, D. C.?
   b. Does your school now provide a course in Mathematics for the Consumer as an elective for students in the 11th and 12th grade? _____
   c. If no, then is your school planning to provide a course in Mathematics for the Consumer?
   d. Do you personally think your school should provide a course in Consumer Mathematics?
7. a. Do you offer a course in Commercial Arithmetic? _____
   If yes, then continue—
   b. Is it taught by someone in the mathematics department? _____
   c. Is it taught by a teacher in the commercial department? _____
   d. What per cent of the high school population take the course? _____
   e. Is the interest of the pupils and parents in commecrial arithmetic increasing _____; or is it decreasing _____?
8. During the war years some schools offered a course for 11th and 12th grade pupils called "Preinduction," or Basic Mathematics.
   a. Did your school provide such a course? _____
   b. Are you now providing such a course? _____
   If yes, then continue—
   c. Is interest in this type of course increasing or is it decreasing? _____
   If no, then continue—
   d. Do you personally think your school should provide such a course? _____
9. Do you think that your school should provide a rigorous course in arithmetic as an elective for 11th and 12th grade students? _____
10. Does your school give any kind of standardized test as a part of the requirements for graduation in an effort to guarantee minimum competence in arithmetic? _____
11. a. Does your school provide a general course in mathematics on the second track, not suggested by the above questions?
    If yes—
    b. What do you call it? _____
    c. Outline a few topics that you cover.
    If no—
    d. Is there a course in mathematics that you feel you should provide? _____
    e. Outline a few topics that you would cover in the new course.
12. If you have any questions that you would like to see in the inquiry be sure to send them with this report. A large number of questions have already been submitted and there is no guarantee that any, except the very best, will be published next year.
13. Feel free to supplement your responses to the preceding inquiry by further comments on separate pages. As stated earlier, the main idea is to picture the present mathematics situation in your school or your building.

When you have filled out the blanks, mail to Raleigh Schorling, Chairman, Commission on Postwar Plans, University High School, Ann Arbor, Michigan.

Concerning Subscriptions that Expire in May

ANY ONE who began his subscription to THE MATHEMATICS TEACHER with the October issue in 1946 is automatically a member of the National Council of Teachers of Mathematics until October, 1947. However, since no issues of the magazine are published in June, July, August and September, those who paid their dues in October of last year should send in their renewals before October, 1947 in order to save the Council inconvenience and loss of money. Costs of publication are rising and in order not to have to raise the price of the journal any further (the present dues are $3) we bespeak the cooperation of our members in being prompt in making renewals. In order to make sure that this matter is not overlooked, THE MATHEMATICS TEACHER sent out cards to all members whose subscriptions expire in May (even though their membership runs to October). It has been almost impossible to plan for the October issue each year because members are so careless about renewing in time. Moreover, entirely too many members fail to renew at all even though two or three notices and a personal appeal from the Editor have been sent out. It is more important now than ever before to "stick by the ship" if we are to weather the storms ahead and if the Council is to continue to do its work effectively. This year we cannot afford to send the October issue to those whose dues have not been previously paid.—EDITOR

Mathematics Institute

Duke University, August 5–15

W. W. RANKIN, *Director*, Duke University
VERYL SCHULT, *Assistant Director*, Washington, D. C.
General Theme—Mathematics at Work
(Junior and Senior High School and College)

PROGRAM

August 5, Tuesday

9 A.M.–1:00 P.M. Registration
2:30–4:00 P.M.

"The Epic of Mathematics" (Mural Paintings)
Miss Clare Leighton, English American Artist
Professor W. W. Rankin, Director of Mural Paintings

Organization of Study Groups
Miss Veryl Schult, Supervisor of Mathematics, Washington, D. C.

8:00–9:30 P.M.

"Some Applications of Elementary Mathematics in Atomic Energy Development Work,"
Dr. Harry Soodak, Physicist, Clinton Laboratories, Oak Ridge, Tenn.

Discussion

August 6, Wednesday

11:30 A.M.–12:30 P.M.

"Some Applications of Mathematics to the Problem of Tire Industry"
Mr. R. D. Evans, Mathematician, Goodyear Tire & Rubber Co., Akron, Ohio

6:30 P.M.

Banquet—Presiding, Dean Herbert Herring, Duke University

"Ways and Means of Closer Cooperation Between Industry and Education"
Mr. R. E. Gillmor, Vice President, Sperry Gyroscope Corporation

"Education and the New Industry"
Dr. Dwayne Orton, Educational Director, International Business Machines Corp.

9:00–10:00 P.M.
Social Hour

August 7, Thursday

11:30 A.M.–12:30 P.M.

"Approach to Study of Research Problems"
Mr. A. F. Underwood, Head Division 5 Mechanical Engineering, General Motors Research Laboratories

Discussion

3:30 P.M.

Visit to Wright Automatic Machinery Co.

8:00–9:45 P.M.

"Mathematics in the Study of Bearings"
Mr. A. F. Underwood, General Motors Laboratories

"Making and Using Films, Instruments, and Models in the Study of Mathematics"
Mr. Roger W. Zinn, Educational Consultant, Jam Handy Organization

Discussion

August 8, Friday

11:15 A.M.–12:30 P.M.

"Dramatic Incidents in the Study of Geometry"
Miss Frances Burns, Oneida High School, Oneida, N. Y.

"Basic Values in Junior High School Mathematics"
Miss Mary C. Rogers, Ch. of Research Committee on Junior H. S. Mathematics, Roosevelt Junior High School, Westfield, N. J.

8:00–10:30 P.M.

Party—University House

"Science, Industry, Mathematics and Standards of Measure"
Dr. Edward U. Condon, Director of Bureau of Standards, Washington, D. C.

August 9, Saturday

11:30 A.M.–12:30 P.M.

"Vital Statistics in Relation to Life Insurance"
Dr. A. J. Lotka, Statistician, Metropolitan Life Insurance Co.

"How to Collect, Organize, and Interpret Data"
(Speaker to be announced)

Discussion

6:00 P.M.—Dinner

"Modern Trends in Mathematical Education"—Movie "Meet North Carolina"
Professor W. D. Reeve, Columbia University

August 10, Sunday

5:00 P.M.

Tea—Rankin Home

August 11, Monday

11:30 A.M.–12:30 P.M.

"Mechanisms and Mathematics"
Mr. Glenn M. Tracey, Chief Research Engineer, Wright Automatic Machinery Co.

Discussion

8:00–9:45 P.M.

"Engineering Methods of Study"
Professor C. R. Vail, Electrical Engineering, Duke University

"Practical Uses of the Limit Concept"

Professor C. G. Mumford, Mathematics, N. C. State College

Discussion

August 12, Tuesday

11:30 A.M.–12:30 P.M.

"The Uses of Probability in Manufacturing"
Mr. D. K. Briggs, Chief of Sounds Instruments Engineering Department, Western Electric Co.

Discussion

Dinner—Carolina Inn, Chapel Hill, N. C.

8:30–10:00 P.M.

"The Uses of Mathematics in Watch Making"
Mr. B. L. Hummel, Hamilton Watch Co., Head of Watch Design Dep't.

Discussion

August 13, Wednesday

11:30 A.M.–12:30 P.M.

"The Mathematics of Some Simple Mechanisms Used in Automatic Packaging Machines"
Mr. John W. May, Chief Engineer, Wright Automatic Machinery Co.

Discussion

2:00 P.M.

Visit to Western Electric Company, Burlington, N. C.

8:00–9:45 P.M.

"The Number System and One to One Correspondence"
Professor F. G. Dressel, Mathematics, Duke University

"Problem Sources and Problem Saving"
Professor J. W. Cell, Mathematics, N. C. State College

Discussion

August 14, Thursday

11:30 A.M.–12:30 P.M.

Panel Discussion—Students—Teachers—Textbooks
Miss Bonnie Cone, Charlotte, N. C.
Miss Veryl Schult, Washington, D. C.
Miss Frances Burns, Oneida, N. Y.

8:00–9:15 P.M.

"The Mathematics of Making and Fitting Lenses"
(Speaker to be announced), Optical Company

9:30 P.M.

Party—Rankin Home (Watermelons)

August 15, Friday

10:30–12:00 Noon

"Curves Used in Engineering"
Professor W. J. Seeley, Head Electrical Engineering, Duke

Study Group Reports
"Q.E.D."—Miss Veryl Schult

12:00 Noon—Adjournment

Registration Fee—$3.00
Room and board—$3.50 per day single room
—$3.00 per day double room

STUDY GROUPS

I—8:30–9:30 Daily

Aids in the Study of Geometry (Junior and Senior High School)
Leader: Miss Frances Burns, Oneida High School, Oneida, N. Y.

II—8:30–9:30 A.M. Daily

Junior High School Mathematics
Leader: Miss Mary C. Rogers, Roosevelt Junior H. S., Westfield, N. J. *Ch.* Senior H. S. Research Com. Association of Mathematics Teachers of New Jersey

III—10:00–11:00 Daily

Making and Using Films, Instruments, Models
Leader: Mr. Roger W. Zinn, Educational Consultant, Jam Handy Organization, Detroit, Michigan

IV—10:00–11:00 Daily

The Enrichment of Mathematics (Junior and Senior H. S. College)
Leader: Miss Veryl Schult, Director of Mathematics, Washington City Schools

V—2:15–3:15 P.M. Daily

Tests and Measurements in Mathematics
Leader: Miss Elinor Douglas, Woodrow Wilson High School, Washington, D. C.

VI—3:30–4:30 P.M. Daily

Applications of Mathematics (High School and College Work)
Leader: Professor J. W. Cell, Mathematics, N. C. State College

VII—2:00–3:15 P.M. Daily

Field Work in Mathematics
Leader:

Programs with detail information will be available after May 1 and reservations may be made after that date

W. W. RANKIN,
Director of Mathematics Institute, Duke University,
Durham, N. C.

Notice to Readers of The Mathematics Teacher

Please do not send money, stamps, or checks to THE MATHEMATICS TEACHER in payment for any of the National Council Yearbooks. All such orders and payments for the same should be sent to *The Bureau of Publications*, Teachers College, Columbia University, 525 W. 120th St., New York 27,, N Y.—EDITOR

Attendance Record of the Twenty-Fifth Annual Meeting

Compiled by Edwin W. Schreiber, *Secretary*,
State Teachers College, Macomb, Illinois

ARKANSAS
Arkadelphia
Foster, Mary Lee
CALIFORNIA
Stanford
Kinney, Lucien
CONNECTICUT
New Britain
Fuller, Kenneth G.
New Haven
Boynton, Holmes
DELAWARE
Wilmington
Green, Ruth Lee
Holladay, Amanda
Loose, Florence
Lynam, Lela
Mulcahy, Helen
DISTRICT OF COLUMBIA
Washington
Brown, Carrie
Brown, Clementine F.
Brown, Cynthia
Derrick, Guinevere
Grubbs, Ethel H.
Haworth, Dr. Ellis
Holland, Gwendolyn
Matyka, Agnes L.
McNaughton, Ella H.
Newton, Beverly
Novinger, Faith F.
Robertson, Helen
Reid, Irene M.
Schult, Veryl
Tolson, Juanita S.
FLORIDA
Clearwater
Morse, C. Marguerite
Gainesville
Kokomoor, F. W.
Miami
Haskins, Fleeta
Ocala
Boss, James
GEORGIA
Atlanta
Patton, Bess
Witcher, Elma
ILLINOIS
Charleston
Heller, Hobart F.
Chicago
Hartung, M. L.
Sister Mary Aquinata
Sister Mary Pulcheria
Evanston
Hildebrandt, E. H. C.
La Grange
Hawkins, George E.
Macomb
Schreiber, Edwin W.
Maywood
Hildebrandt, Martha
INDIANA
East Chicago
Kauffman, Geraldine
Muncie
Whitcraft, L. H.
West Lafayette
Carnahan, Walter H.
IOWA
Cedar Falls
Trimble, Harold C.
Iowa City
Price, Vernon
LOUISIANA
Baton Rouge
Karnes, Houston T.
MARYLAND
Baltimore
Atkins, Joseph K.
Blackiston, Nanette
Bowers, Eunice
Clishiam, Mildred
Culley, Alfred E.
Davis, Elaine C.
Deppenbrock, Audrey
Gardner, Elizabeth
Gardner, Elinora
Heinzerling, Margaret
Herbert, Agnes
Hodges, Norma L.
Holzapfel, Gertrude
Howard, James R.
Kresslein, Kathleen
Lohrfink, Emma
McHale, Helen
McPherson, Eva
McPherson, Helen
Mebane, Joseph
Meyers, Shirley
Norris, Grover
Ortmann, Arnold
Reese, Sara M.
Schwartz, Edna F.
Schwartz, J. Karl
Scott, Carrie D.
Shank, Ella
Shattuck, Mary
Shaw, Catherine
Taylor, Dorothy
Baltimore
Taylor, S. LeRoy
Taylor, Margaret
Tillinghast, Richard
Ziefle, Lydia
Cumberland
Hamilton, Margaret
Frederick
Brown, Lillian
Hagerstown
Coffman, Janet
Raspebury
Donat, V. J.
Salisbury
Russell, May
MASSACHUSETTS
Arlington
Smith, Jessie
Boston
Garland, Harold B.
Gifford, Anna S.
Meder, Elsa M.
Syer, Henry W.
Chelsea
Share, Nathaniel
Fall River
Macdonald, M. E.
Martin, Grace A.
Longmeadow
Smith, Rolland R.
Swampscott
Betts, Barbara
MICHIGAN
Ann Arbor
Jones, Phillip
Karpinski, L. C.
Schorling, Raleigh
Detroit
Thiele, C. L.
Royal Oak
Zinn, W. Roger
Kalamazoo
Myers, Shelden S.
MINNESOTA
Alexandria
Norskog, Edna M.
Minneapolis
Woolsey, Edith
Johnson, Donavan
Brueckner, Leo J.
St. Cloud
Smith, Jessie R.
MISSOURI
Columbia
Leonhardy, Adele
Kansas City
Young, Mary C.
Kirksville
Jamison, G. H.
St. Louis
Marth, Ella
NEW HAMPSHIRE
Exeter
Adkins, Jackson
Plymouth
Smith, Geneva M.
NEW JERSEY
Atlantic City
Barton, Zella
Dukes, Mary
Foren, Harriet
Freed, Herbert
Gillingham, Samuel
Ingersoll, Neva
Kelly, May J.
Molinari, Dorthy
Nelson, Wilma
Spangler, Lloyd
Sr. M. Casimir
Sr. M. DeSales
Stanberg, Bertha A.
Steinberg, Elsie
Werntz, Mrs. C. L.
Whelan, Dorothy
Belleville
Hefferman, John
Belvidere
Sterner, William
Wieghorst, Jane
Bloomfield
Andrews, Helen B.
Dunn, Margaret
Wilson, Virna

Boonton
Tompkins, Sidney
Bridgeton
Gibson, Julia S.
Harris, Edwin W.
Joslin, Vera
VanKeuren, Margaret
Caldwell
Sharpe, Lewis
Clementon
Edinger, David
Clifton
Jackson, Mildred Mrs.
Jackson, Mildred
Collingswood
Brinker, Myretta
Colts Neck
Freer, Marian
East Orange
Nordgaard, Martin
Fellows, Harold A.
Risinger, H. B.
Robinson, Fannie H.
Elizabeth
Cullen, Elizabeth
Gorgens, Florence
Loughren, Amanda
Glassboro
McCabb, Robert
Glen Rock
Palmer, Mrs. Bertha
Haddon Heights
Nagle, Sylvania
High Bridge
Bogart, Myra
Daugherty, Gladys
Highland Park
Fulcher, Lois
Hamilton, William H.
Hightstown
Rogers, Grace N.
Hoboken
Housman, Ida E.
Jackson, Arthur
Jersey City
Bedford, Fred
McMackin, Faith
Sanders, Henry
Kearney
Gibson, Euphemia
Lawrenceville
Sturley, Erio
Linden
Reynolds, Elizabeth
Rose, Agnes
Hersh, Beatrice
Madison
Lockhart, Roy
Starkey, S. Herbert
Millburn
Vincentz, Audrey
Millville
Coombs, Elizabeth
Montclair
Clifford, Paul C.
Fehr, Howard F.
Mallory, Virgil S.
Reid, Wallace E.
Moorestown
McKelvey, Eileen
Newark
Ayers, N. Howard
Cleary, Catherine
Frost, Gladys
Humphreys, Mildred
Newark
Littauer, Sebastian
Lumb, Leonard
Reid, Virginia
New Brunswick
Fender, Fred
Rickenbrode, Francyl
Waring, Agnes L.
North Arlington
Wydeman, Jane
Wydeman, Esther
Nutley
Assmus, Edward F.
Grossnickle, Foster
Harrison, Henrietta
Paterson
Daugherty, J. Dwight
Offhouse, Charles D.
Pemberton
Delzp, John M.
Perth Amboy
Kertes, Ferdinand
Plainfield
Garthwaite, George
Smith, Arthur
Ridgewood
Benedict, Carroll (Mr.)
Carter, Frances
Miller, Ralph C.
Powell, Nell
Roselle
Messner, Madeline
Rutherford
Black, C. W.
Higgins, William
Sayreville
Hudak, Irene
Kreiger, Grace E.
Walsh, Maragret
Wnukowski, Lucian
Scotch Plains
Olson, Ellen
Roberts, Dorothy
South Orange
Aeschbach, Margaret
Woodard, Mary Ann
Swedesboro
Stump, Sarah
Summit
Emerson, Josephine
Teaneck
Venn, Lidie
Trenton
Cleary, Margaret
Jeannette, Agnes
Berger, Doris
Daniels, Dorothy
Daniels, Edith
Delate, Edward J.
Dibinseppe, Josephine
Kistrup, Nora
Lentini, Fred
Marsh, William L.
Nelson, L. Wayne
Notaro, Antoinette
Sanowitz, Sheila
Shuster, Carl N.
Sinclair, Warren F.
Smith, Alan M.
Union City
Campen, George
Drake, John
Korbett, Frank
Zofay, Agnes
Veneland
Flamingo, Margaret
Ventnor
Dawson, Sara B.
Foulke, Eliza J.
Rhodes, Mable V.
Westfield
Baldwin, Ruth I.
Rogers, Mary C.

NEW YORK

Alfred
Seidlin, Joseph
Brooklyn
Bergstresser, C. A.
Jokiel, Margaret
Lazar, Nathan
Ross, George J.
Schutzman, Wm.
Bronx
Butler, Lovilla
Feinstein, Anita
Hausle, Eugenie
Canastota
Daniels, Gertrude
Cortland
Sueltz, Ben A.
Endicott
Smith, Elizabeth
Floral Park
Levensaler, Nina
Anderson, Alfred T.
Clark, John J.
Ostrander, Ida
Flushing
Wise, Henry
Garden City
Baravalle, H. V.
Heinecke, Elmer J.
Rhoad, Daniel
Hempstead
Windt, Alfred
Huntington
Van Wart, Edna
Johnson City
Courtright, Anne
New York City
Austin, Sarah
Baogley, Wm.
Barcus, Howard
Braverman, Ben.
Brown, Harold
Clark, John
Foran, Francis
Gardner, Randolph
Johnson, Esther
Nilsen, Lorna
Noland, Florence
Reeve, Katherine
Reeve, W. D.
Rosenberg, Beatrice
Schlauch, Williams
Secor, May
Smith, Esther
Steinbrenner, Arthur
Turner, Ethel M.
Wahlert, Howard E.
Oceanside
Rapp, Elaine
Carlson, Florence
Pettit, Ida
Oneonta
Johnson, Frances
Sanford, Vera
Burns, Frances
Richfield Springs
Wharton, Naomi
Rochester
Betz, William
Sayville
Hertlin, Lillian
Hirzel, Leona

Staten Island
Landau, Dorothy
Syracuse
Stokes, Ruth W.
Valley Stream
Henry, Etta M.
Herkstroter, Merle
Woodmere
Wood, Florence
Yonkers
Wickham, Joseph

NORTH CAROLINA
Durham
Rankin, W. A.
Greenville
Brown, Kenneth

OHIO
Cincinnati
Sr. M. Reilly
Sr. M. Joseph
Cleveland
Grime, Herschel
MacLearie, James A.
Miller, A. Brown
Columbus
Fawcett, Harold
Hetherington, Louise
Kinsella, John J.
Shaker Heights
Miller, Florence
Oberlin
Newsom, Carroll
Oxford
Christofferson, H. C.

PENNSYLVANIA
Abington
Woodruff, J. S.
Bryn Mawr
Williamson, Helen
Bucks County
Fraser, Grant
Dallas
Davis, Cornelia
Bucks County
Evans, Walter
Hatboro
Smith, Leonard L.
Havertown
Collins, Miriam
Indiana
Mahachek, Joy
Schnell, Leroy
Jenkintown
Vansant, Helen E.
Lansdowne
Morris, Esther
Lovelto
Mino, Rev. P.
Millersville
Boyer, Lee
Monessen
Williams, E. H.
Russell, Margaret
Montoursville
Price, Ralph W.
Perhasi
Miller, Martin L.
Philadelphia
Darnell, Alice H.
Hurley, Thomas
Reilly, Margaret
Breininger, Herman
Kneedler, Harry
Metzner, William
Philadelphia
Stein, Edwin I.
Stokes, C. Newton
Wade, Francis
White, Robert
Pittsburgh
Sr. M. Michael
Sr. M. Tarcisius
Whitman, Edwin A.
Stroudsburg
Lee, Harold
Lee, Mildred
May, Jonas T.
Sieg, Julia
Swarthmore
Brinkmann, H. W.
Titusville
Wertz, Halfred
Trucksville
Saxe, Esther
Upper Darby
Trescott, B. M.
Wayne
Lutz, Rose E.
Waynesboro
Andrews, Irene
West Chester
Pickett, Hale
Wrightsville
Cromie, Ruth
York
Schriver, Harold

SOUTH CAROLINA
Anderson
O'Neal, Alice B.

TENNESSEE
Nashville
Wren, F. Lynwood

TEXAS
Marshall
Heard, Mrs. Ida

VIRGINIA
Falls Church
Rumbough, William
Forest
Taylor, Ruth

WEST VIRGINIA
Charlottsville
Lankford, Francis
St. Albans
Lynch, Kathryn

WISCONSIN
Racine
Potter, Mary

HONOLULU
Annis, Lilla Lee

CANADA
Newmarket
Rourke, Robert
Toronto
Colgrove, R. G.

ATTENDANCE BY STATES

Arkansas	1
California	1
Connecticut	2
Delaware	5
District of Columbia	15
Florida	4
Georgia	2
Illinois	8
Indiana	3
Iowa	2
Louisiana	1
Maryland	40
Massachusetts	10
Michigan	6
Minnesota	5
Missouri	4
New Hampshire	2
New Jersey	132
New York	50
North Carolina	2
Ohio	11
Pennsylvania	43
South Carolina	1
Tennessee	1
Virginia	2
West Virginia	2
Wisconsin	1
Honolulu	1
Canada	2
TOTAL	369

VOTE BY STATES, ANNUAL ELECTION FEBRUARY 1947.
COMPILED BY EDWIN W. SCHREIBER, SECRETARY

Ala.	5	Iowa	11	N.H.	4	S.D.	3
Ariz.	3	Kans.	27	N.J.	41	Tenn.	9
Ark.	6	Ky.	9	Nev.	0	Tex.	23
Calif.	36	La.	9	N.M.	0	Utah	0
Colo.	10	Me.	3	N.Y.	57	Vt.	4
Conn.	13	Md.	17	N.C.	6	Va.	9
Dela.	3	Mass.	31	N.D.	1	Wash.	5
D. of C.	17	Mich.	39	Ohio	48	W.Va.	3
Fla.	4	Minn.	12	Okla.	18	Wis.	21
Ga.	3	Mo.	23	Ore.	6	Wyo.	3
Idaho	0	Miss.	4	Penna.	40		
Ill.	118	Mont.	1	R.I.	4	Total	748
Ind.	22	Neb.	9	S.C.	8		

Report of the Treasurer

EDWIN W. SCHREIBER, State Teachers College,
Macomb, Illinois

For the Year, Feb. 1, 1946 to Feb. 1, 1947

Balance on hand at beginning of year:			
Union National Bank of Macomb	$ 852.17		
Savings Bank Deposit	3,842.15		
U. S. Treas. Bond, 2%, June 15, 1952–54	1,000.00		
U. S. Treas. Bond, 2½%, Mar. 15, 1966–71	4,000.00		
U. S. Treas. Bond, 2½%, Dec. 15, 1967–72	1,000.00		$10,694.32
Receipts for the year:			
Mathematics Teacher, W. D. Reeve	1,843.52		
Bureau of Publications, Yearbooks	0.00		
Interest on Bonds	147.05		
Interest on Savings	23.80		2,014.37
			$12,708.69
Expenditures for the year:			
6th Cleveland Meeting		$ 756.94	
Commission on Post-war Plans		389.41	
President's Office		200.00	
Secretary-Treasurer's Office:			
Postage and Supplies	175.00		
Printing	50.00		
Stationery	75.00		
Ballot	150.00		
Secretary Service	300.00		
Banking Expense	3.80	753.80	
Committee on Co-operation with A.A.A.S.		122.28	
Committee on Multi-sensory Aids		25.00	
Sectional Meetings		140.00	
State Representatives		300.00	
Com. on Co-operation with Other Organizations		98.36	
Contingency		86.30	2,872.09
			$ 9,836.60

Statement of Assets in the Treasurer's Office

Commercial Bank Deposit	$ 781.08		
Savings Bank Deposit	3,055.52		
U. S. Treas Bond, 2%	1,000.00		
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The above report has been audited and found correct.

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For further information write to:

Harold Fawcett
Department of Education
Ohio State University
Columbus 10, Ohio

BOSTON UNIVERSITY
GENERAL COLLEGE

178 NEWBURY STREET
BOSTON 16, MASSACHUSETTS

Dear Co-Worker:

A revision of the Encyclopaedia of Educational Research has been scheduled, and I am taking an interest in the article on arithmetic. Can you find time to re-read that article and then give me your frank suggestions?

The following will provide a rough organization for your suggestions:

1. Research lines not covered in the article —Research references for the same
2. Research studies, along lines developed in the article, that are not mentioned— References for the same
3. Suggestions for re-arrangement of article
4. Suggestions for different emphasis in article
5. Other suggestions

Since I may need to see some of the new researches, I have arranged with Miss Eleanor Herdman, Librarian of the School of Education, 84 Exeter St., Boston, to co-operate in any interlibrary loans that may be necessary.

May I hear from you soon, with a promise of help and an approximate date on which I may expect further word from you?

Thanking you and appreciating your cooperation, I am

Sincerely yours,

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